IMPRS
Plasma Physics I
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Magnetohydrodynamics

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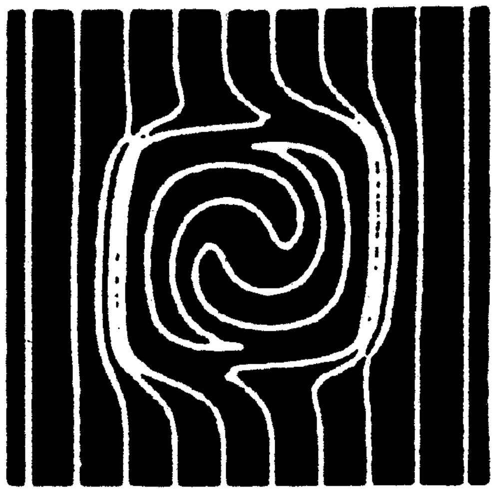


MHD equations – Summary

$$c\boldsymbol{\nabla}\times\boldsymbol{B} = 4\pi\boldsymbol{j}\ , \quad c\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}\ , \quad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

$\lambda\boldsymbol{j} = \boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}$, λ electrical resistivity, λ^{-1} electrical conductivity

$$\rho\frac{\partial\boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}p + \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} + \text{external forces}$$

$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0$, $\quad p = f(\rho, T)$, $\quad$ energy equation

Basic assumptions:

- $v \ll c$: system stationary on light travel time, no em waves
- high electrical conductivity: $\boldsymbol{E}$ determined by $\partial\boldsymbol{B}/\partial t$, not by charges σ

$c\frac{E}{L} \approx \frac{B}{T} \curvearrowright \frac{E}{B} \approx \frac{1}{c}\frac{L}{T} \approx \frac{v}{c} \ll 1$, E plays minor role : $\frac{e_{el}}{e_m} \approx \frac{E^2}{B^2} \ll 1$

$\frac{\partial\boldsymbol{E}/\partial t}{c\boldsymbol{\nabla}\times\boldsymbol{B}} \approx \frac{E/T}{cB/L} \approx \frac{E}{B}\frac{v}{c} \approx \frac{v^2}{c^2} \ll 1$, displacement current negligible

pre-Maxwell equations Galilei-covariant

same for heuristic Ohm's law, valid for collisional plasma, $\boldsymbol{j} \nparallel \boldsymbol{v}$

charge density σ given by $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\sigma$

Lorentz force $\frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} = \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = -\nabla\frac{B^2}{8\pi} + \frac{1}{4\pi}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}$

magnetic pressure, magnetic tension

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial P_{ik}}{\partial x_k} + \text{external forces}$$

$$P_{ik} = \rho v_i v_k + p\delta_{ik} + \frac{B^2}{8\pi}\delta_{ik} - \frac{1}{4\pi}B_i B_k$$

Reynolds stress, Maxwell's stress

external forces: e.g. gravity $\rho\boldsymbol{g}$, viscous friction $\rho\nu\nabla^2\boldsymbol{v}$

equation of state: perfect gas $p = R\rho T$

energy equation : T prescribed, adiabacy $\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\frac{p}{\rho^\gamma} = 0$

Various simplifications:

$\boldsymbol{v}$ given $\curvearrowright$ magnetokinematics

$\boldsymbol{v} = 0$ $\curvearrowright$ magnetohydrostatics

$\partial/\mathrm{d}t = 0$ $\curvearrowright$ stationarity

linearized equations $\curvearrowright$ waves, stability

5. Magnetohydrokinematics

evolution of $\boldsymbol{B}$ under influence of given $\boldsymbol{v}$

5.1 Induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = c\nabla\times\boldsymbol{E} = -c\nabla\times\left(\lambda\boldsymbol{j} - \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right) = -c\nabla\times\left(\frac{\lambda c}{4\pi}\nabla\times\boldsymbol{B} - \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)$$

$$= \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) - \nabla\times\left(\frac{\lambda c^2}{4\pi}\nabla\times\boldsymbol{B}\right) = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) - \eta\nabla\times\nabla\times\boldsymbol{B}$$

with $\eta = \frac{\lambda c^2}{4\pi} = \text{const}$ magnetic diffusivity

induction, diffusion

$$\nabla\times(\boldsymbol{v}\times\boldsymbol{B}) = -\boldsymbol{B}\,\nabla\cdot\boldsymbol{v} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{B}$$

expansion/contraction, shear/stretching, advection

5.2 Free Decay

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\eta \nabla \times \nabla \times \boldsymbol{B}$$

$$\boldsymbol{B} = (B_x(y,t), 0, 0) \;, \quad \frac{\partial B_x}{\partial t} = \eta \frac{\partial^2 B_x}{\partial y^2} \;, \quad B_x(0) = B_x(L) = 0$$

$$B_x(y,t) = \sum_n c_n \sin\left(\frac{n\pi y}{L}\right) \exp\left(-\frac{n^2\pi^2 t}{\tau}\right) \text{ with } \tau = \frac{L^2}{\eta}$$

decay modes, $l_n \approx L/n$, $\tau_n \approx l_n^2/\eta \approx \tau/n^2$

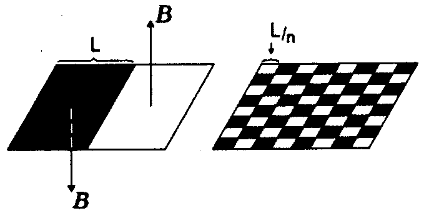
decay of magnetic field in homogenous sphere

	Earth	Sun	sunspots
L [cm]	10^8	10^{11}	10^9
η [cm^{2}s^{-1}]	10^5	10^4	10^{11}
τ [y]	$3 \cdot 10^4$	10^{11}	1

turbulent diffusion

$\eta_t \sim v_t l_t \sim 10^5 \cdot 10^8 = 10^{13}$ cm^{2}s^{-1}

Sun $\tau_t \sim 30\ y$



5.3 Alfven theorem

ideal conductor $\eta = 0$: $\quad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B})$

magnetic flux through floating surface is constant : $\quad \frac{d}{dt}\int_F \boldsymbol{B} \cdot d\boldsymbol{F} = 0$

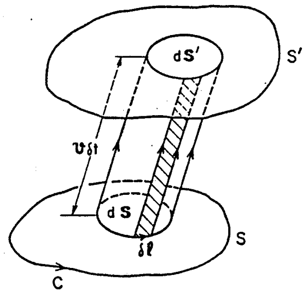
proof:

$$0 = \int \nabla \cdot \boldsymbol{B} dV = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{F} = \int_F \boldsymbol{B}(t) \cdot d\boldsymbol{F} - \int_{F'} \boldsymbol{B}(t) \cdot d\boldsymbol{F}' - \oint_C \boldsymbol{B}(t) \cdot d\boldsymbol{s} \times \boldsymbol{v}\, dt$$

$$\int_{F'} \boldsymbol{B}(t+dt) \cdot d\boldsymbol{F}' - \int_F \boldsymbol{B}(t) \cdot d\boldsymbol{F} = \int_F \{\boldsymbol{B}(t+dt) - \boldsymbol{B}(t)\} \cdot \mathrm{d}\boldsymbol{F} - \oint_C \boldsymbol{B} \cdot d\boldsymbol{s} \times \boldsymbol{v}\, dt$$

$$= dt \left(\int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{F} - \oint_C \boldsymbol{B} \cdot d\boldsymbol{s} \times \boldsymbol{v} \right) = dt \left(\int \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{F} - \oint_C \boldsymbol{B} \cdot d\boldsymbol{s} \times \boldsymbol{v} \right)$$

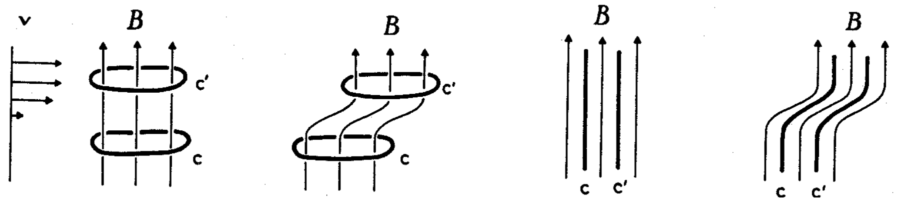
$$= dt \left(\oint_C \boldsymbol{v} \times \boldsymbol{B} \cdot d\boldsymbol{s} - \oint_C \boldsymbol{B} \cdot d\boldsymbol{s} \times \boldsymbol{v} \right) = 0$$



frozen-in field lines

impression that magnetic field follows flow

but $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$ and $\nabla \times \boldsymbol{E} = -c\partial \boldsymbol{B}/\partial t$



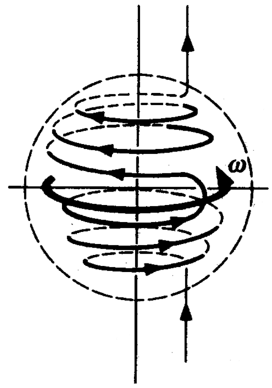
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = -\boldsymbol{B}\,\nabla\cdot\boldsymbol{v} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{B}$$

(i) star contraction

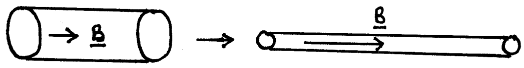
$\overline{B} \sim R^2$, $\overline{\rho} \sim R^{-3} \curvearrowright \overline{B} \sim \overline{\rho}^{2/3}$

Sun $\curvearrowright$ white dwarf $\curvearrowright$ neutron star

ρ [g cm^{-3}]: $1 \curvearrowright 10^6 \curvearrowright 10^{15}$



(ii) stretching of flux tube:



Bd^2 = const, ld^2 = const $\curvearrowright B \sim l$

(iii) shear, differential rotation:

$\partial B_\phi / \partial t = r \sin\theta\, \nabla\Omega \cdot \boldsymbol{B}_p$

5.4 Walen equation

combine ideal induction equation and continuity equation

$$\frac{d}{dt}\left(\frac{\boldsymbol{B}}{\rho}\right) = \frac{\boldsymbol{B}}{\rho}\cdot\nabla\boldsymbol{v}\,, \quad \text{integrated} \quad \frac{\boldsymbol{B}}{\rho} = \frac{\boldsymbol{B}_0}{\rho_0}\cdot\nabla_0\boldsymbol{r}$$

5.5 Magnetic Reynolds number

dimensionless variables: length L, velocity v_0, time L/v_0

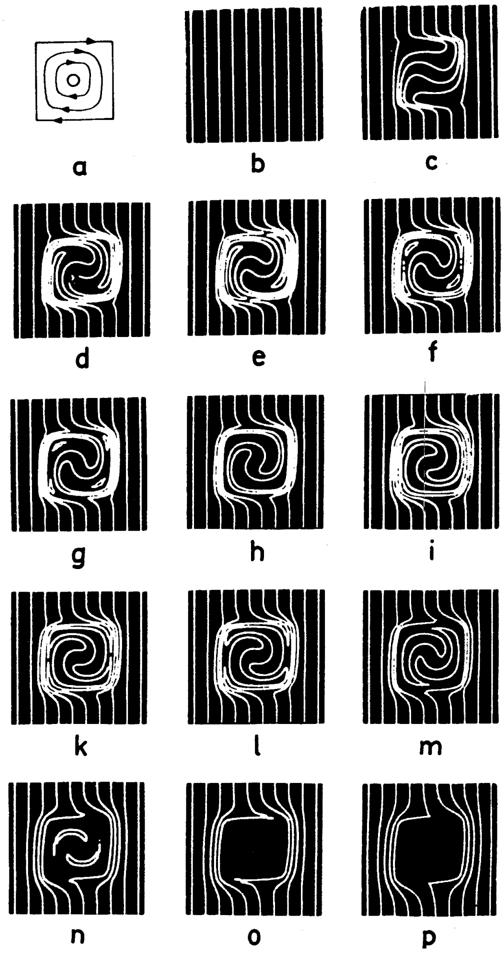
$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - R_m^{-1}\,\nabla\times\nabla\times\boldsymbol{B} \quad \text{with} \quad R_m = \frac{v_0 L}{\eta}$$

as combined parameter

laboratorium: $R_m \ll 1$, cosmos: $R_m \gg 1$

induction for $R_m \gg 1$, diffusion for $R_m \ll 1$, e.g. for small L

flux expulsion from closed velocity fields



6. MHD equilibria

6.1 Lorentz force

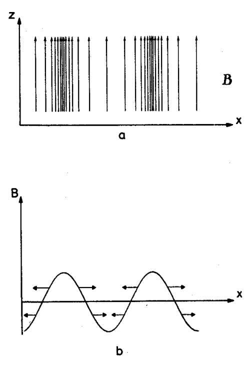
$$\frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} = \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = -\nabla\frac{B^2}{8\pi} + \frac{1}{4\pi}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}$$

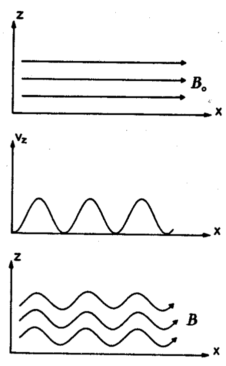
magnetic pressure

$$\boldsymbol{B} = (0, 0, B(x)) , \quad \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{1}{8\pi}\left(\frac{dB^2}{dx}, 0, 0\right)$$

magnetic tension

$$\boldsymbol{B} = (B_0, 0, v_z' B_0 dt) , \quad \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \frac{1}{4\pi}\left(0, 0, v_z'' B_0^2 dt\right)$$





6.2 Magnetohydrostatics

$$-\nabla p + \frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} \ \ (+\rho\boldsymbol{g}) \ = 0\,, \quad \nabla\cdot\boldsymbol{B} = 0$$

plasma beta $\beta = \dfrac{p}{B^2/8\pi}$

6.2.1 Plasma cylinder

(i) driven by constant current along cylinder

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial\theta} = 0\,, \quad j_z = \text{const} \quad \curvearrowright \quad B_r = B_z = 0$$

$$\frac{\partial}{\partial r}\left(p + \frac{B_\theta^2}{8\pi}\right) + \frac{B_\theta^2}{4\pi r} = 0 \quad \text{and} \quad \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = \frac{4\pi}{c}j_z = \text{const}$$

$$B_\theta = \frac{2\pi}{c}j_z r \ \text{ inside}, \quad B_\theta = \frac{2\pi}{c}j_z\frac{a^2}{r} \ \text{ outside}, \quad p = p_0 - \frac{\pi j_z^2 r^2}{c^2} \ \text{ inside}$$

(ii) driven by axial surface current only

$$p_0 = \frac{B_0^2}{8\pi} \ \text{ on surface } \ r = a$$

$$B_\theta = 0 \ \text{ inside}, \quad B_\theta = \frac{C}{r} \ \text{ outside}$$

6.2.2 Force-free fields

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = 0\,, \quad \nabla\times\boldsymbol{B} = \mu\boldsymbol{B}\,, \quad 0 = \nabla\cdot\mu\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\mu$$

μ constant along field lines

(i) μ = const: cylindrical coordinates, cylind. and axial symmetry

$$-\frac{\partial B_z}{\partial r} = \mu B_\theta\,, \ \frac{1}{r}\frac{\partial}{\partial r}rB_\theta = \mu B_z \ \curvearrowright \ B_z = B_0 J_0(\mu r)\,, \ B_\theta = B_0 J_1(\mu r)$$

(ii) $\mu \neq$ const: magnetic energy density $F(r) = \left(B_\theta^2 + B_z^2\right)/8\pi$

as generating function

$$B_\theta^2 = -4\pi r\frac{dF}{dr}\,, \ B_z^2 = 8\pi\left(F + \frac{1}{2}r\frac{dF}{dr}\right) \ \curvearrowright \ \frac{dF}{dr} \leq 0\,, \ F + \frac{1}{2}r\frac{dF}{dr} \geq 0$$

exercise : $F = \dfrac{B_0^2}{8\pi}\dfrac{1}{a^2 + r^2} \quad \curvearrowright \quad B_\theta, B_z, \mu$

6.2.3 Magnetohydrostatics of a prominence

cool, dense clouds in solar corona along inversion lines

$$-\nabla p + \frac{1}{4\pi}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \rho \boldsymbol{g} = 0 \,, \quad \nabla \cdot \boldsymbol{B} = 0 \,, \quad p = R\rho T$$

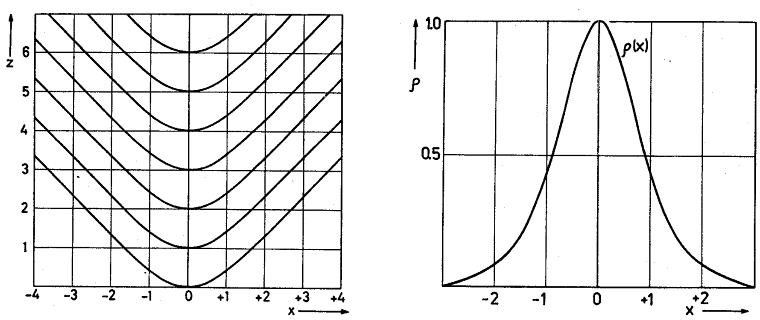
$$\boldsymbol{B} = (B_x, 0, B_z) \,, \quad \boldsymbol{g} = (0, 0, -g) \,, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0 \,, \quad T = \text{const}$$

$$\nabla \cdot \boldsymbol{B} = 0 \ \curvearrowright \ B_x = \text{const}$$

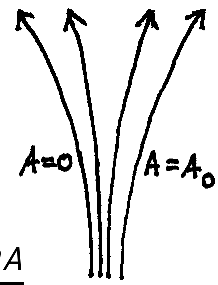
$$RT \frac{\partial \rho}{\partial x} = -\frac{1}{4\pi} B_z \frac{\partial B_z}{\partial x} \,, \quad 0 = \frac{1}{4\pi} B_x \frac{\partial B_z}{\partial x} - \rho g$$

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\alpha}{2} \frac{\partial B_z^2}{\partial x} \quad \text{with} \quad \alpha^{-1} = \frac{RT}{g} B_x = \text{const}$$

$$B_z = B_z^\infty \tanh\left(\frac{1}{2}\alpha B_z^\infty x\right) \quad \text{and} \quad \rho = \frac{1}{8\pi RT}\left(\frac{B_z^\infty}{\cosh \frac{1}{2}\alpha B_z^\infty x}\right)^2$$



6.2.4 Flux tubes



$$-\nabla p + \frac{1}{4\pi}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \rho g \boldsymbol{e}_z = 0 \,, \quad \nabla \cdot \boldsymbol{B} = 0$$

$$p = R\rho T \,, \quad \frac{\partial}{\partial y} = 0 \,, \quad B_y = 0$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \,, \quad \boldsymbol{A} = (0, A, 0) \,, \quad B_x = -\frac{\partial A}{\partial z} \,, \quad B_z = \frac{\partial A}{\partial x}$$

$$\frac{B_x}{B_z} = \frac{dx}{dz} \,, \quad dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial z} dz = 0 \,, \quad A = \text{const.} \quad \text{field lines}$$

$$\frac{\partial p}{\partial x} + \frac{1}{4\pi} \nabla^2 A \frac{\partial A}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} + \frac{1}{4\pi} \nabla^2 A \frac{\partial A}{\partial z} + \rho g = 0$$

$$(x, z) \longrightarrow (A, z)$$

$$\left(\frac{\partial p}{\partial A}\right)_z + \frac{1}{4\pi} \nabla^2 A = 0 \quad \text{and} \quad \left(\frac{\partial p}{\partial z}\right)_A + \rho g = 0$$

2nd eq. hydrostatics along field lines

$$\curvearrowright \quad p(A, z) = p_0(A) \exp\left\{-\int \frac{g}{RT(A, z)} dz\right\} \,, \quad \rho(A, z)$$

1st eq. Grad – Shafranov equation for $A(x, z)$

quite difficult, esp. because of boundaries, $p + \dfrac{B^2}{8\pi} = p_e$

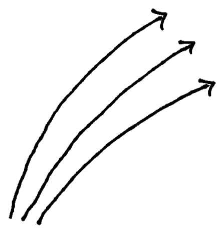
$$\nabla^2 A = \frac{1}{2}\left(\frac{\partial B^2}{\partial A}\right)_z - \left(\frac{\partial B_x}{\partial z}\right)_A \,, \quad \frac{\partial}{\partial A}\left(p + \frac{B^2}{8\pi}\right)_z = \frac{1}{4\pi}\left(\frac{\partial B_x}{\partial z}\right)_A$$

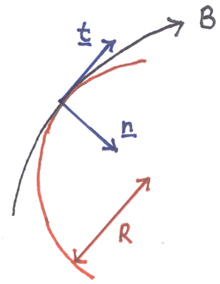
$$p + \frac{B^2}{8\pi} = \frac{1}{4\pi}\int\left(\frac{\partial B_x}{\partial z}\right)_A dA + p_e(z)$$

$$\int\left(\frac{\partial B_x}{\partial z}\right)_A dA = 0 \quad \text{at boundaries}$$

thin flux tubes $\quad \ldots = 0 \quad$ everywhere

6.2.5 Thin flux tubes





$\boldsymbol{r} = \boldsymbol{r}(s)\,, \quad s$ arc – length

$\hat{\boldsymbol{t}} = \frac{\partial \boldsymbol{r}}{\partial s}\,, \quad \hat{\boldsymbol{n}} = R\frac{\partial \hat{\boldsymbol{t}}}{\partial s}\,, \quad \hat{\boldsymbol{b}} = \hat{\boldsymbol{t}} \times \hat{\boldsymbol{n}}$ unit vectors, R curvature radius

Serret – Frenet for $\tau = 0$: $\quad \frac{\partial \hat{\boldsymbol{n}}}{\partial s} = -R\hat{\boldsymbol{t}}\,, \quad \frac{\partial \hat{\boldsymbol{b}}}{\partial s} = 0\,, \quad {}' = \frac{\partial}{\partial x}$

$$ds = \sqrt{dx^2 + dz^2}\,, \; R = \frac{(1+z'^2)^{3/2}}{z''}\,, \; \boldsymbol{e}_z \cdot \hat{\boldsymbol{t}} = \frac{z'}{\sqrt{1+z'^2}}\,, \; \boldsymbol{e}_z \cdot \hat{\boldsymbol{n}} = \frac{1}{\sqrt{1+z'^2}}$$

$p + \frac{B^2}{8\pi} = p_e$ lateral pressure balance

$-\frac{\partial p_e}{\partial z} - \rho g = 0$ external stratification

$$-\boldsymbol{\nabla} p + \frac{1}{4\pi}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} - \rho g \boldsymbol{e}_z \quad | \cdot\hat{\boldsymbol{t}},\ \cdot\hat{\boldsymbol{n}},\ \cdot\hat{\boldsymbol{b}}$$

$$-\frac{\partial p}{\partial s} - \rho g(\boldsymbol{e}_z\cdot\hat{\boldsymbol{t}}) = 0 \quad \curvearrowright \quad -\frac{\partial p}{\partial z} - \rho g = 0$$

$$-(\rho-\rho_e)g(\boldsymbol{e}_z\cdot\hat{\boldsymbol{n}}) + \frac{B^2}{4\pi R} = 0 \quad \curvearrowright \quad -(\rho-\rho_e)g + \frac{z''}{1+z'^2}\frac{B^2}{4\pi} = 0$$

$-(\rho-\rho_e)g(\boldsymbol{e}_z\cdot\hat{\boldsymbol{b}}) = 0$ tube in plane $\perp \hat{\boldsymbol{b}}$ for $\rho \neq \rho_e$

$T(z),\ T_e(z)$ given

$p(z),\ p_e(z)$ from tangential component and external stratification

$B^2/8\pi$ from lateral pressure balance

$2\frac{z''}{1+z'^2} = -\frac{g(\rho-\rho_e)}{p-p_e} = \frac{d/dz(p_e-p)}{p_e-p}$ from normal component

$x \rightarrow z$ independent variable, $z' = u$ dependent variable

$$z'' = \frac{du}{dx} = \frac{du}{dz}\frac{dz}{du} = u\frac{du}{dz} = \frac{1}{2}\frac{du^2}{z} = \frac{1}{2}\frac{dz'^2}{dz}$$

$$\frac{dz'^2/dz}{1+z'^2} = \frac{d}{dz}\log(p_e-p)\ , \quad \log(1+z'^2) = \log(p_e-p)$$

$z' = \sqrt{C(p_e-p)-1}\ ,\quad z(x)$ by quadrature

6.3 Magnetohydrodynamics of stellar winds

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial\phi} = \frac{\partial}{\partial\theta} = 0\ , \quad \boldsymbol{v}(r) = (v_r, 0, v_\phi)\ , \quad \boldsymbol{B}(r) = (B_r, 0, B_\phi)$$

$\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0 \quad \curvearrowright \quad r^2\rho v_r = \text{const} = I$

$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \quad \curvearrowright \quad r^2 B_r = \text{const} = \Phi$

$\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) = 0 \quad \curvearrowright \quad r(v_\phi B_r - v_r B_\phi) = \text{const} = \Omega\Phi \quad (*)$

$$\rho(\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}) = -\boldsymbol{\nabla} p + \frac{1}{4\pi}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} - \rho g\boldsymbol{e}_z$$

ϕ–component $\quad \curvearrowright \quad rv_\phi - \frac{\Phi}{4\pi I} rB_\phi = \text{const} = D \quad (**)$

$(*)$ and $(**)$ $\quad \curvearrowright \quad rv_\phi = \frac{M_A^2 D - \Omega r^2}{M_A^2 - 1}$ with

$M_A^2 = \frac{4\pi I^2}{\rho\Phi^2} = \frac{v_r^2}{B_r^2/8\pi} = \left(\frac{v_r}{v_A}\right)^2$ Alfven Mach number

$M_A < 1$ at Sun, $M_A > 1$ at Earth

$M_A = 1$ at $r = r_A$ Alfven radius $\sim 10\ldots 20\ R_\odot \quad \curvearrowright \quad D = r_A^2\Omega$

$$rv_\phi = r_A^2\Omega\frac{M_A^2 - (r/r_A)^2}{M_A^2 - 1} \quad \rightarrow \quad r_A^2\Omega \quad \text{for} \quad r\rightarrow\infty$$

$$\frac{B_r}{B_\phi} = \frac{M_A^2 - 1}{M_A^2(r_A^2 - r^2)\Omega} r v_\phi < 0 \quad \rightarrow \quad 0 \quad \text{for} \quad r \rightarrow \infty$$

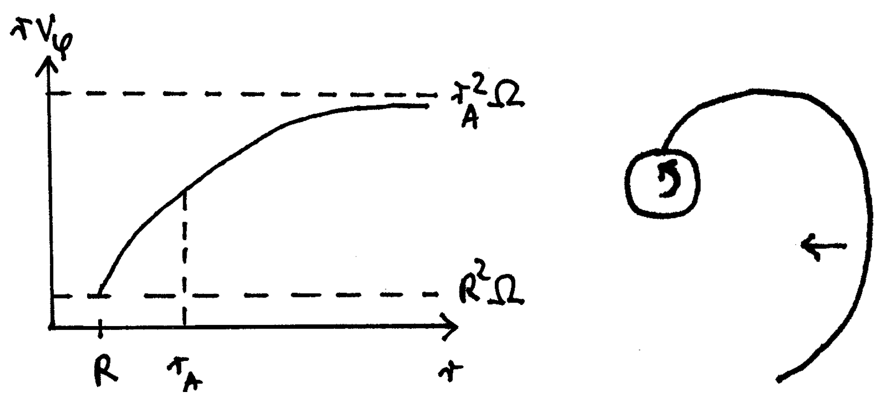
angular momentum per unit mass at ∞: $\quad r_A^2\Omega$

radial wind under rigid guidance until r_A

conservation of angular momentum ↷ magnetic braking

$$(M + \Delta M)R^2\Omega = (MR^2 + \Delta M r_A^2)(\Omega - \Delta\Omega)$$

$$\frac{\Delta\Omega}{\Omega} = \frac{\Delta M}{M}\left(\frac{r_A^2}{R^2} - 1\right) \quad \text{or} \quad \frac{\dot{\Omega}}{\Omega} = \frac{\dot{M}}{M}\left(\frac{r_A^2}{R^2} - 1\right)$$



7. MHD waves

perturbation of a (static) equilibrium ↷ instability, waves

instability: complicated configurations

now waves: simple equilibrium, e.g. homogenous medium

7.1 Linearisation

$$\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x}, t)\ ,\ \boldsymbol{v} = \boldsymbol{v}_1(\boldsymbol{x}, t)\ ,\ p = p_0 + p_1(\boldsymbol{x}, t)\ ,\ \rho = \rho_0 + \rho_1(\boldsymbol{x}, t)$$

index 0: equilibrium, static, homogenous, satisfies MHD eqs.

$\boldsymbol{B}_0$ = const, $\boldsymbol{j}_0 = 0$, $\boldsymbol{v}_0 = 0$, p_0 = const, ρ_0 = const

index 1: perturbations $\ll$ equilibrium

adiabatic perturbations $p_1 = c_s^2\rho_1$ with $c_s^2 = \gamma p_0/\rho_0$ = const

c_s adiabatic sound speed

$$\frac{\partial\rho_1}{\partial t} + \rho_0\nabla\cdot\boldsymbol{v}_1 = 0\ ,\ \frac{\partial\boldsymbol{B}_1}{\partial t} = \nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_0)$$

$$\rho_0\frac{\partial\boldsymbol{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{4\pi}(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_0$$

elimination of p_1, ρ_1 and $\boldsymbol{B}_1$ in favour of $\boldsymbol{v}_1$

$$\frac{\partial^2 \boldsymbol{v}_1}{\partial t^2} = c_s^2 \nabla(\nabla\cdot\boldsymbol{v}_1) + [\nabla\times\{\nabla\times(\boldsymbol{v}_1\times\boldsymbol{v}_A)\}]\times\boldsymbol{v}_A$$

$\boldsymbol{v}_A = \boldsymbol{B}_0/\sqrt{4\pi\rho_0}$ Alfvén velocity

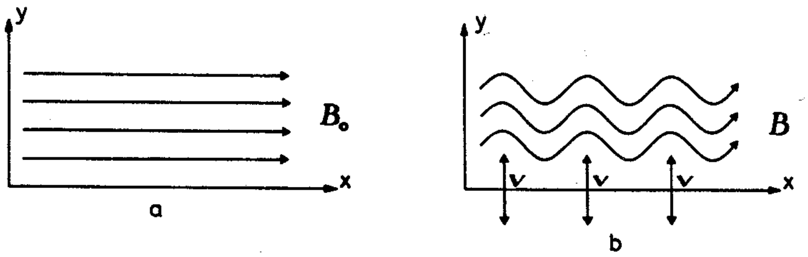
linear, homogenous, constant coefficients, $\boldsymbol{v}_1 \sim \exp[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)]$

$$\omega^2 \boldsymbol{v}_1 = \left(c_s^2 + v_A^2\right)(\boldsymbol{k}\cdot\boldsymbol{v}_1)\,\boldsymbol{k} + \boldsymbol{v}_A\cdot\boldsymbol{k}\,[(\boldsymbol{v}_A\cdot\boldsymbol{k})\,\boldsymbol{v}_1 - (\boldsymbol{v}_A\cdot\boldsymbol{v}_1)\,\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{v}_1)\,\boldsymbol{v}_A]$$

7.2 Alfvén waves

$\boldsymbol{v}_1 \perp (\boldsymbol{v}_A, \boldsymbol{k}) \quad \curvearrowright \quad \omega = \boldsymbol{v}_A\cdot\boldsymbol{k} = \pm v_A k \cos\theta$ and $\nabla_{\boldsymbol{k}}\omega = \boldsymbol{v}_A$

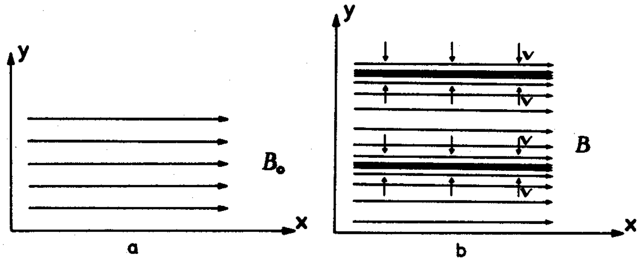
transversal, dispersion free, magnetic tension



7.3 Compressional waves

$\boldsymbol{k} \parallel \boldsymbol{v}_1 \perp \boldsymbol{v}_A \perp \boldsymbol{k} \quad \curvearrowright \quad \omega/k = \sqrt{c_s^2 + v_A^2}$

longitudinal, dispersion free, pressure and magnetic pressure gradient



7.4 Waves inclined to magnetic field

$\sphericalangle(\boldsymbol{v}_A, \boldsymbol{k}) = \theta$

dispersion relation vector equation ↷ three scalar equations

vanishing secular determinant ↷ three ω^2 for each $\boldsymbol{k}$

one solution is the Alfvén wave

the other two are the fast and slow magnetoacoustic waves

$$v_{ph}^2 = \begin{cases} (v_A \cos\theta)^2 \\ \frac{1}{2}\left(c_s^2 + v_A^2\right) \pm \frac{1}{2}\left[\left(c_s^2 + v_A^2\right)^2 - 4c_s^2 v_A^2 \cos^2\theta\right]^{1/2} \end{cases}$$

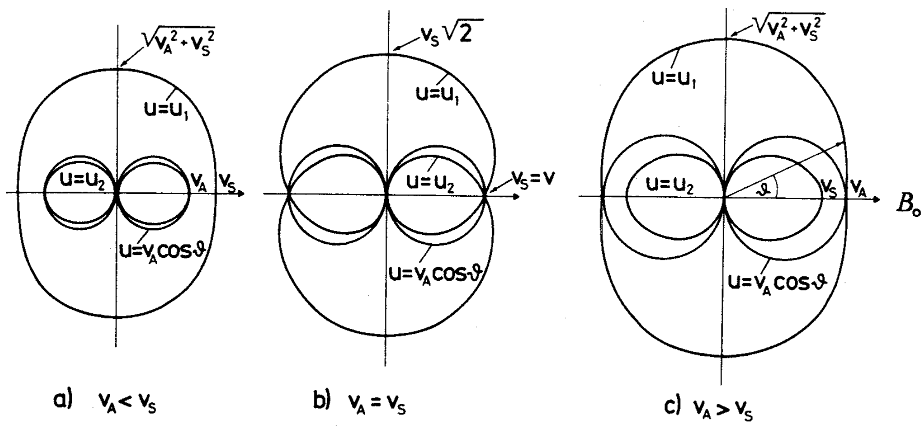
pressure and magnetic restoring forces roughly in phase

↷ fast waves

roughly out of phase ↷ slow waves

displacements of three modes make a triad of orthogonal vectors

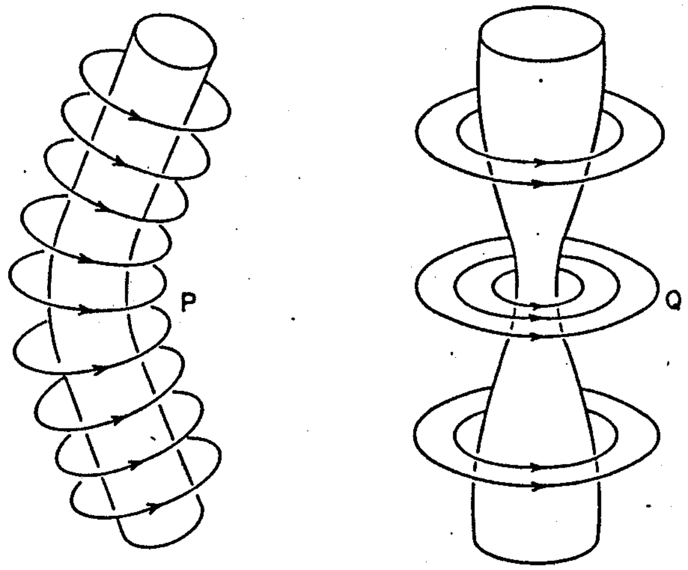
↷ any disturbance superposition of Alfvén, fast and slow modes



$u_1 > \max(v_A, v_s)$, $u_2 < \min(v_A, v_s)$

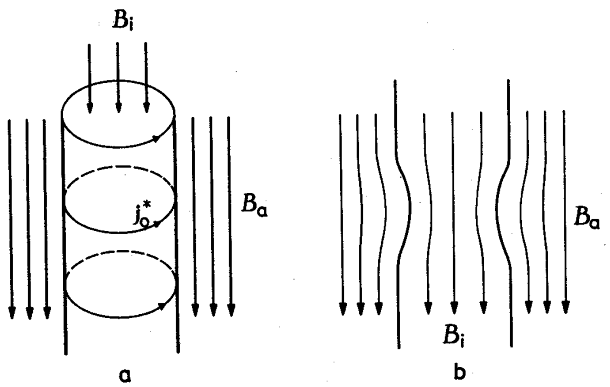
8. MHD stability

8.1 Plasma cylinder with toroidal field outside

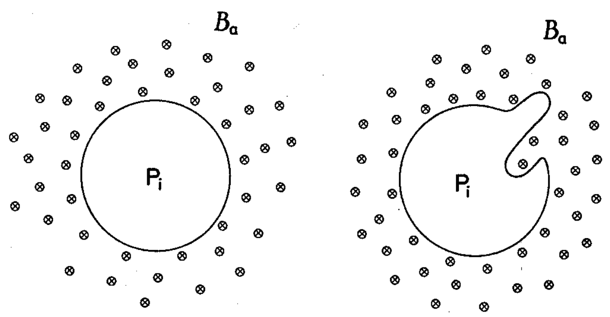


$B^2_{\text{left}}/8\pi < p_0 < B^2_{\text{right}}/8\pi$ $\qquad$ $p_0 < B^2_{\text{waist}}/8\pi$

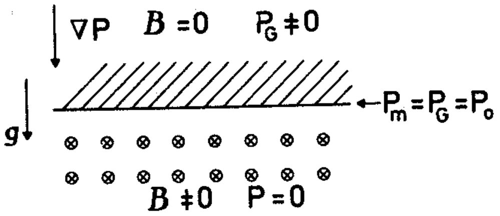
8.2 Plasma cylinder with homog. field outside

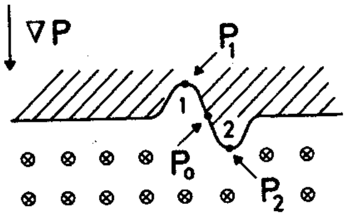


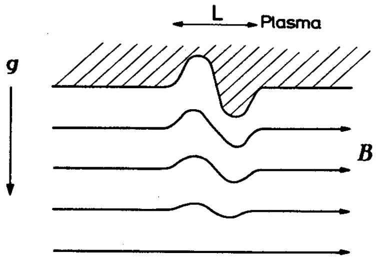
$p_i + B_i^2/8\pi = B_e^2/8\pi$ $\qquad$ $B^2_{e,1} < B^2_{e,0}$ and $B^2_{i,1} > B^2_{i,0}$



8.3 Rayleigh-Taylor instability







8.4 Normal modes analysis

static equilibrium, small adiabatic perturbations

time-dependence of perturbations

formalism similar to waves, but equilibrium spatial-dependent

$$\rho_0 \frac{\partial^2 \boldsymbol{v}_1}{\partial t^2} = \boldsymbol{L}\boldsymbol{v}_1 \quad \text{with}$$

$$\boldsymbol{L}\boldsymbol{v}_1 = \nabla[(\boldsymbol{v}_1 \cdot \nabla) p_0 + \gamma p_0 \nabla \cdot \boldsymbol{v}_1] + \frac{1}{4\pi}(\nabla \times \boldsymbol{B}_0) \times [\nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0)]$$
$$+ \frac{1}{4\pi} \nabla \times \nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) \times \boldsymbol{B}_0 - (\nabla \cdot \rho_0 \boldsymbol{v}_1)\boldsymbol{g}$$

real, linear, homogenous, time – independent differential operator

$\boldsymbol{v}_1 = \hat{\boldsymbol{v}}_1(\boldsymbol{x}) \exp i\varpi t \quad \curvearrowright \quad -\rho_0 \varpi^2 \hat{\boldsymbol{v}}_1 = \boldsymbol{L}\hat{\boldsymbol{v}}_1$

$\varpi_I = \Im(\varpi)$ determines stability

$\varpi_I < 0 \curvearrowright \boldsymbol{v}_1$ grows exponentially $\curvearrowright$ instability

boundary conditions: $\boldsymbol{v}_1 \cdot \boldsymbol{n} = 0$, $\boldsymbol{B} \cdot \boldsymbol{n} = 0$, $\boldsymbol{E} \times \boldsymbol{n} = 0$

or infinite system with $\boldsymbol{v}_1$, $\boldsymbol{B}_1$, ... $\rightarrow 0$ for $|\boldsymbol{x}| \rightarrow \infty$

operator $\boldsymbol{L}$ self-adjoint: $\int \boldsymbol{u}\cdot\boldsymbol{L}\boldsymbol{v}\,dV = \int \boldsymbol{v}\cdot\boldsymbol{L}\boldsymbol{u}\,dV$

$\curvearrowright$ eigenvalues ϖ_k^2 real, no overstability
eigenfunctions $\boldsymbol{v}_k$ orthogonal and complete

$(\varpi_k^2, \boldsymbol{v}_k)$ normal modes

$\forall\varpi_k^2 \geq 0 \Leftrightarrow$ stability, oscillation or wave

$\exists\varpi_k^2 < 0 \Leftrightarrow$ monotonous instability

lower bound to ϖ^2 : $\quad -\varpi^2 = \dfrac{\int \boldsymbol{v}_1\cdot\boldsymbol{L}\boldsymbol{v}_1 dV}{\int \rho_0 \boldsymbol{v}_1^2 dV}$ variational form

find $\boldsymbol{v}_1$ which maximizes $\int \boldsymbol{v}_1\cdot\boldsymbol{L}\boldsymbol{v}_1 dV$

example: hydrodynamic Rayleigh-Taylor instability

$$\frac{d}{dz}\left(\rho_0\varpi^2\frac{dv_{1z}}{dz}\right) = k^2\left(\rho_0\varpi^2 - g\frac{d\rho_0}{dz}\right)v_{1z} \quad \text{eigenvalue equation}$$

$$-\varpi^2 = \int \frac{d\rho_0}{dz} g v_{1z}^2 dz \left\{\int \rho_0 \left[\frac{1}{k^2}\left(\frac{dv_{1z}}{dz}\right)^2 + v_{1z}^2\right] dz\right\}^{-1} \text{incompressible}$$

if $\mathrm{d}\rho_0/dz > 0$ somewhere $\curvearrowright \varpi^2 < 0 \curvearrowright$ instability

compressible Rayleigh-Taylor instability

$$-\varpi^2 = \int \left(\frac{d\rho_0}{dz} + \frac{\rho^2 g}{\gamma p_0}\right) g v_{1z}^2 dz \left\{\int \rho_0 \left[\frac{1}{k^2}\left(\frac{dv_{1z}}{dz}\right)^2 + v_{1z}^2\right] dz\right\}^{-1}$$

Schwarzschild's convection criterium

magnetic Rayleigh-Taylor instability, incompressible

$$-\varpi^2 = \int \left\{\frac{d\rho_0}{dz} g v_{1z}^2 - (\boldsymbol{k}\cdot\boldsymbol{B}_0)^2 \left[\frac{1}{k^2}\left(\frac{dv_{1z}}{dz}\right)^2 + v_{1z}^2\right]\right\} dz \left\{\int \rho_0 \left[\frac{1}{k^2}\left(\frac{dv_{1z}}{dz}\right)^2 + v_{1z}^2\right] dz\right\}^{-1}$$

8.5 Small displacements and energy integral

displacement $\boldsymbol{\xi}$ such that $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{\xi}$, $\boldsymbol{v}_1(\boldsymbol{x}) = \partial\boldsymbol{\xi}/\partial t|_{\boldsymbol{x}_0} = \boldsymbol{v}_1(\boldsymbol{x}_0)$

$$\rho_0 \frac{\partial^2 \boldsymbol{v}_1}{\partial t^2} = \boldsymbol{L}\boldsymbol{v}_1 \quad \curvearrowright \quad \rho_0 \frac{\partial^3 \boldsymbol{\xi}}{\partial t^3} = \boldsymbol{L}\frac{\partial \boldsymbol{\xi}}{\partial t} \quad \curvearrowright \quad \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \boldsymbol{L}\boldsymbol{\xi}$$

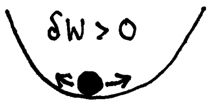
multiplication by $\partial\boldsymbol{\xi}/\partial t$ and integration, $\boldsymbol{L}$ self-adjoint:

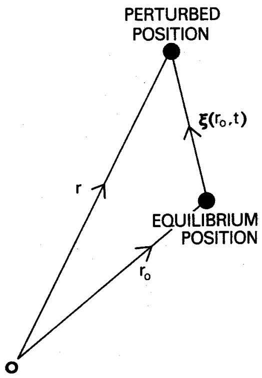
$$\frac{\partial}{\partial t}\frac{1}{2}\int \rho_0 \left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right)^2 dV = \int \rho_0 \frac{\partial \boldsymbol{\xi}}{\partial t}\cdot\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} dV = \int \frac{\partial \boldsymbol{\xi}}{\partial t}\cdot\boldsymbol{L}\boldsymbol{\xi}\, dV = \frac{\partial}{\partial t}\frac{1}{2}\int \boldsymbol{\xi}\cdot\boldsymbol{L}\boldsymbol{\xi}\, dV$$

$$\frac{1}{2}\int \rho_0 \left(\frac{\partial \boldsymbol{\xi}}{\partial t}\right)^2 dV = \frac{1}{2}\int \boldsymbol{\xi}\cdot\boldsymbol{L}\boldsymbol{\xi}\, dV\,, \quad \delta K = -\delta W$$

$\forall\boldsymbol{\xi}$: $\delta W > 0$ $\curvearrowright$ stability

$\exists\boldsymbol{\xi}$: $\delta W < 0$ $\curvearrowright$ instability





minimization of δW with respect to all admissible $\boldsymbol{\xi}$

8.6 Energy method

$$\delta W = -\frac{1}{2}\int \boldsymbol{\xi}\cdot\Big\{\nabla\left[\boldsymbol{\xi}\cdot\nabla p_0 + \gamma p_0 \nabla\cdot\boldsymbol{\xi}\right] + \frac{1}{4\pi}(\nabla\times\boldsymbol{B}_0)\times[\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)]$$
$$+\frac{1}{4\pi}\nabla\times\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)\times\boldsymbol{B}_0 - (\nabla\cdot\rho_0\boldsymbol{\xi})\boldsymbol{g}\Big\}dV$$

$$\boldsymbol{a}\cdot\nabla\lambda = \nabla\cdot(\lambda\boldsymbol{a}) - \lambda\nabla\cdot\boldsymbol{a}\,,\quad \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b}) = -\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) + \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a})$$

$$\int \nabla\cdot\boldsymbol{a}\, dV = \int \boldsymbol{a}\cdot d\boldsymbol{F}$$

$$\delta W = \frac{1}{2}\int \Big\{[\boldsymbol{\xi}\cdot\nabla p_0 + \gamma p_0\nabla\cdot\boldsymbol{\xi}]\nabla\cdot\boldsymbol{\xi} - \frac{1}{4\pi}\boldsymbol{\xi}\cdot[(\nabla\times\boldsymbol{B}_0)\times(\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0))]$$
$$+\frac{1}{4\pi}[\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)]^2 + (\nabla\cdot\rho\boldsymbol{\xi})\boldsymbol{g}\Big\}dV$$
$$-\frac{1}{2}\int \Big\{\boldsymbol{\xi}[\boldsymbol{\xi}\cdot\nabla p_0 + \gamma p_0\nabla\cdot\boldsymbol{\xi}] + \frac{1}{4\pi}[(\boldsymbol{\xi}\times\boldsymbol{B}_0)\times(\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}_0))]\Big\}\cdot d\boldsymbol{F}$$

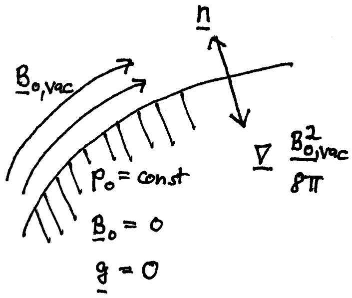
2nd and 4th term of volume integral positive definite $\curvearrowright$ stabilizing acoustic, Alfvén and fast magnetoacoustic waves

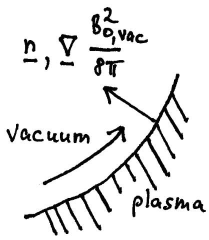
1st, 3rd and 5th term: either positive or negative, if negative potentially hydrodynamic and current-driven instabilities surface integral vanishes for rigid boundaries $\boldsymbol{\xi} = 0$

surface integral on plasma-vacuum boundary:

$$-\frac{1}{2}\int \ldots \cdot d\boldsymbol{F} = -\frac{1}{2}\int (\boldsymbol{\xi}\cdot\boldsymbol{n})^2\, \boldsymbol{n}\cdot\boldsymbol{\nabla}\left(p_0 + \frac{B_0^2}{8\pi} - \frac{B_{0,\mathrm{vac}}^2}{8\pi}\right) dF$$
$$+ \int_{\mathrm{vac}} \frac{1}{8\pi}\left[\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)\right]^2 dV$$

stability determined by sign of $\boldsymbol{n}\cdot\boldsymbol{\nabla}(\ldots)$, independent of $\boldsymbol{\xi}$





$$\delta W = \frac{1}{2}\int \gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 dV + \frac{1}{2}\int (\boldsymbol{\xi}\cdot\boldsymbol{n})^2\, \boldsymbol{n}\cdot\boldsymbol{\nabla}\frac{B_{0,\mathrm{vac}}^2}{8\pi} dF + \int_{\mathrm{vac}} \frac{1}{8\pi}\left[\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)\right]^2 dV$$

$$> 0 \qquad\qquad \gtrless 0 \qquad\qquad > 0$$

concave confinement destabilizing, convex confinem. stabilizing

fluting instability

8.7 Stationary equilibrium

$$-\rho_0\omega^2\boldsymbol{\xi} + 2\omega i\boldsymbol{A}\boldsymbol{\xi} + \boldsymbol{B}\boldsymbol{\xi} = 0$$
$$-\omega^2\int \rho_0\boldsymbol{\xi}^2 dV + 2\omega\int i\boldsymbol{\xi}\cdot\boldsymbol{A}\boldsymbol{\xi}\, dV + \int \boldsymbol{\xi}\cdot\boldsymbol{B}\boldsymbol{\xi}\, dV = 0$$
$$-\omega^2 a + 2\omega b + c = 0$$

$i\boldsymbol{A}$ Hermitian, $\boldsymbol{B}$ self-adjoint, integrals a, b, c real

stability if $c > 0$ or $b^2 + ac > 0$

potentially overstability