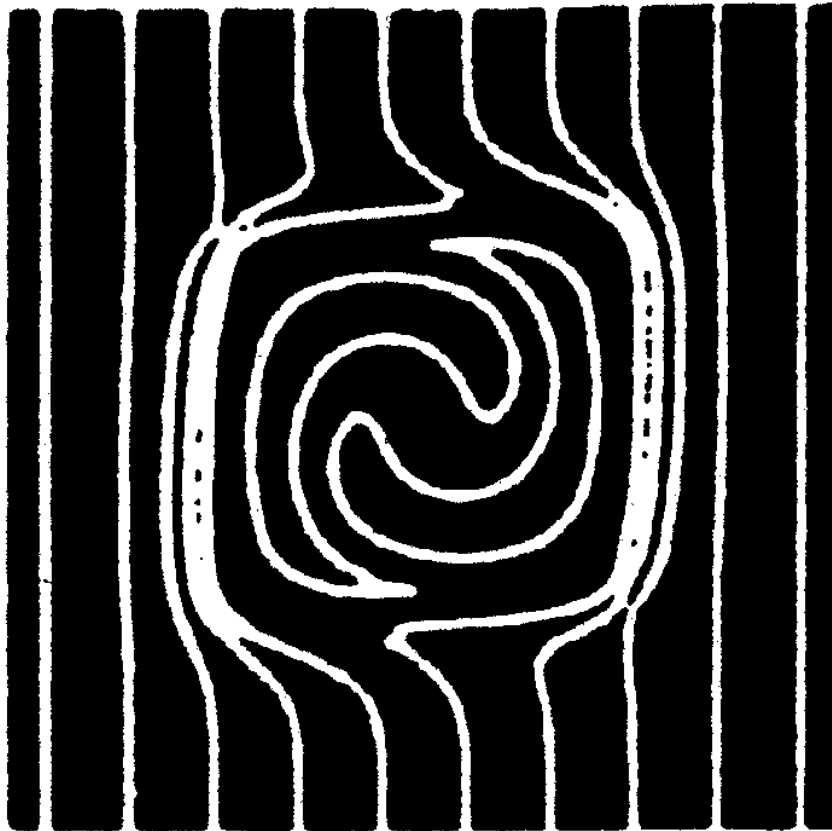


IMPRS
Plasma Physics I
June 2002

Magnetohydrodynamics

Dieter Schmitt (Katlenburg-Lindau)



IMPRS, 6/2002

MHD

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MHD equations – Summary

$$c\nabla \times \mathbf{B} = 4\pi \mathbf{j}, \quad c\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0$$

$$\lambda \mathbf{j} = \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}, \quad \lambda \text{ electrical resistivity, } \lambda^{-1} \text{ electrical conductivity}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \rho + \frac{1}{c} \mathbf{j} \times \mathbf{B} + \text{external forces}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad p = f(\rho, T), \quad \text{energy equation}$$

Basic assumptions:

- $v \ll c$: system stationary on light travel time, no em waves
- high electrical conductivity: \mathbf{E} determined by $\partial \mathbf{B} / \partial t$, not by charges σ

$$c \frac{E}{L} \approx \frac{B}{T} \leadsto \frac{E}{B} \approx \frac{1L}{cT} \approx \frac{v}{c} \ll 1, \quad E \text{ plays minor role: } \frac{e_{el}}{e_m} \approx \frac{E^2}{B^2} \ll 1$$

$$\frac{\partial E / \partial t}{c \nabla \times \mathbf{B}} \approx \frac{E/T}{cB/L} \approx \frac{Ev}{Bc} \approx \frac{v^2}{c^2} \ll 1, \quad \text{displacement current negligible}$$

pre-Maxwell equations Galilei-covariant

same for heuristic Ohm's law, valid for collisional plasma, $\mathbf{j} \parallel \mathbf{v}$

charge density σ given by $\nabla \cdot \mathbf{E} = 4\pi\sigma$

$$\text{Lorentz force } \frac{1}{c} \mathbf{j} \times \mathbf{B} = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \frac{B^2}{8\pi} + \frac{1}{4\pi} (\mathbf{B} \cdot \nabla) \mathbf{B}$$

magnetic pressure, magnetic tension

$$\frac{\partial \rho v_i}{\partial t} = -\frac{\partial P_{ik}}{\partial x_k} + \text{external forces}$$

$$P_{ik} = \rho v_i v_k + p \delta_{ik} + \frac{B^2}{8\pi} \delta_{ik} - \frac{1}{4\pi} B_i B_k$$

Reynolds stress, Maxwell's stress

external forces: e.g. gravity $\rho \mathbf{g}$, viscous friction $\rho \nu \nabla^2 \mathbf{v}$

equation of state: perfect gas $p = R \rho T$

$$\text{energy equation : } T \text{ prescribed, adiabacy } \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \frac{p}{\rho \gamma} = 0$$

Various simplifications:

\mathbf{v} given \leadsto magnetokinematics

$\mathbf{v} = 0 \leadsto$ magnetohydrostatics

$\partial/dt = 0 \leadsto$ stationarity

linearized equations \leadsto waves, stability

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5. Magnetohydrokinematics

evolution of B under influence of given v

5.1 Induction equation

$$\begin{aligned}\frac{\partial B}{\partial t} &= c \nabla \times E = -c \nabla \times \left(\lambda j - \frac{1}{c} v \times B \right) = -c \nabla \times \left(\frac{\lambda c}{4\pi} \nabla \times B - \frac{1}{c} v \times B \right) \\ &= \nabla \times (v \times B) - \nabla \times \left(\frac{\lambda c^2}{4\pi} \nabla \times B \right) = \nabla \times (v \times B) - \eta \nabla \times \nabla \times B\end{aligned}$$

with $\eta = \frac{\lambda c^2}{4\pi} = \text{const}$ magnetic diffusivity

induction, diffusion

$$\nabla \times (v \times B) = -B \nabla \cdot v + (B \cdot \nabla) v - (v \cdot \nabla) B$$

expansion/contraction, shear/stretching, advection

5.2 Free Decay

$$\frac{\partial \mathbf{B}}{\partial t} = -\eta \nabla \times \nabla \times \mathbf{B}$$

$$\mathbf{B} = (B_x(y, t), 0, 0), \quad \frac{\partial B_x}{\partial t} = \eta \frac{\partial^2 B_x}{\partial y^2}, \quad B_x(0) = B_x(L) = 0$$

$$B_x(y, t) = \sum_n c_n \sin\left(\frac{n\pi y}{L}\right) \exp\left(-\frac{n^2 \pi^2 t}{\tau}\right) \quad \text{with } \tau = \frac{L^2}{\eta}$$

decay modes, $l_n \approx L/n$, $\tau_n \approx l_n^2/\eta \approx \tau/n^2$

decay of magnetic field in homogenous sphere

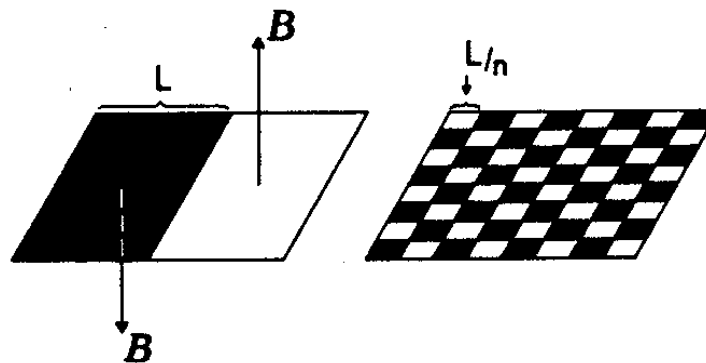
Earth Sun sunspots

L [cm]	10^8	10^{11}	10^9
η [cm^2s^{-1}]	10^5	10^4	10^{11}
τ [y]	$3 \cdot 10^4$	10^{11}	1

turbulent diffusion

$$\eta_t \sim v_t l_t \sim 10^5 \cdot 10^8 = 10^{13} \text{ cm}^2\text{s}^{-1}$$

$$\text{Sun } \tau_t \sim 30 \text{ y}$$



5.3 Alfven theorem

ideal conductor $\eta = 0$: $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$

magnetic flux through floating surface is constant : $\frac{d}{dt} \int_F \mathbf{B} \cdot d\mathbf{F} = 0$

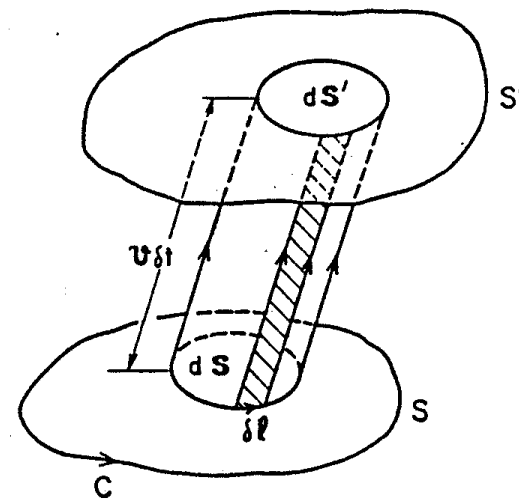
proof:

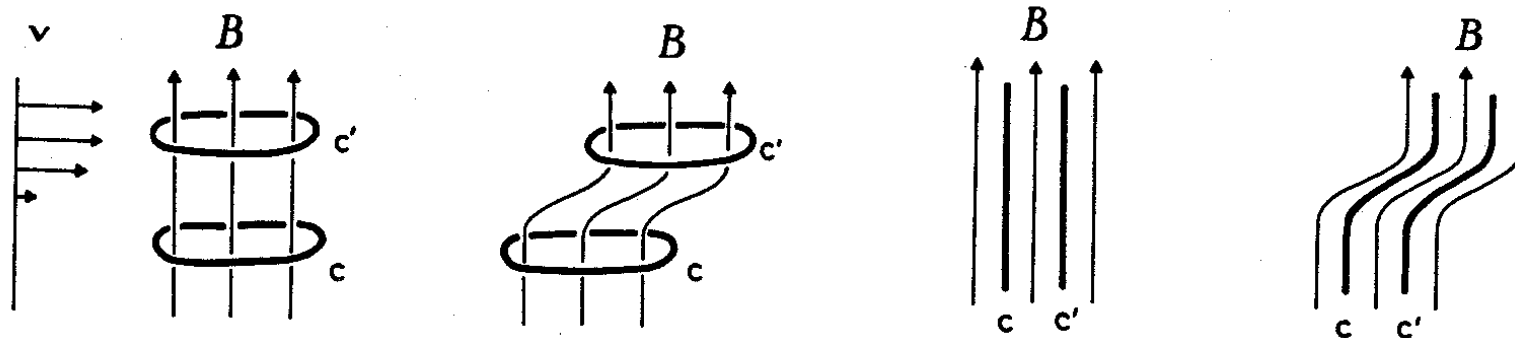
$$\begin{aligned}
 0 &= \int \nabla \cdot \mathbf{B} dV = \int \mathbf{B} \cdot d\mathbf{F} = \int_F \mathbf{B}(t) \cdot d\mathbf{F} - \int_{F'} \mathbf{B}(t) \cdot d\mathbf{F}' - \oint_C \mathbf{B}(t) \cdot d\mathbf{s} \times \mathbf{v} dt \\
 &\int_{F'} \mathbf{B}(t+dt) \cdot d\mathbf{F}' - \int_F \mathbf{B}(t) \cdot d\mathbf{F} = \int_F \{ \mathbf{B}(t+dt) - \mathbf{B}(t) \} \cdot d\mathbf{F} - \oint_C \mathbf{B} \cdot d\mathbf{s} \times \mathbf{v} dt \\
 &= dt \left(\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{F} - \oint_C \mathbf{B} \cdot d\mathbf{s} \times \mathbf{v} \right) = dt \left(\int \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{F} - \oint_C \mathbf{B} \cdot d\mathbf{s} \times \mathbf{v} \right) \\
 &= dt \left(\oint_C \mathbf{v} \times \mathbf{B} \cdot d\mathbf{s} - \oint_C \mathbf{B} \cdot d\mathbf{s} \times \mathbf{v} \right) = 0
 \end{aligned}$$

frozen-in field lines

impression that magnetic field follows flow

but $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ and $\nabla \times \mathbf{E} = -c \partial \mathbf{B} / \partial t$





$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) = -\mathbf{B} \nabla \cdot \mathbf{v} + (\mathbf{B} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{B}$$

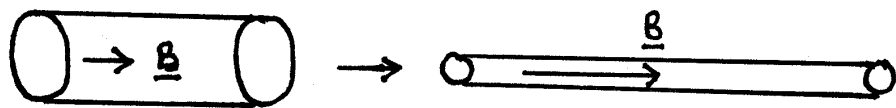
(i) star contraction

$$\bar{B} \sim R^2, \bar{\rho} \sim R^{-3} \leadsto \bar{B} \sim \bar{\rho}^{2/3}$$

Sun \leadsto white dwarf \leadsto neutron star

$$\rho \text{ [g cm}^{-3}\text{]}: 1 \leadsto 10^6 \leadsto 10^{15}$$

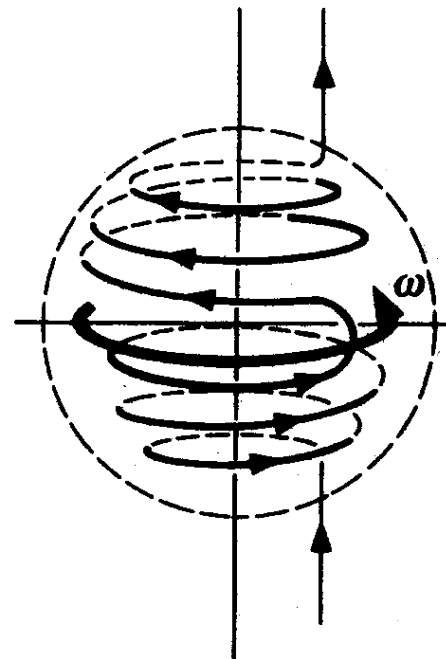
(ii) stretching of flux tube:



$$Bd^2 = \text{const}, l'd^2 = \text{const} \leadsto B \sim l$$

(iii) shear, differential rotation:

$$\frac{\partial B_\phi}{\partial t} = r \sin \theta \nabla \Omega \cdot \mathbf{B}_p$$



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5.4 Walen equation

combine ideal induction equation and continuity equation

$$\frac{d}{dt} \left(\frac{\mathbf{B}}{\rho} \right) = \frac{\mathbf{B}}{\rho} \cdot \nabla \mathbf{v} , \quad \text{integrated} \quad \frac{\mathbf{B}}{\rho} = \frac{\mathbf{B}_0}{\rho_0} \cdot \nabla_0 r$$

5.5 Magnetic Reynolds number

dimensionless variables: length L , velocity v_0 , time L/v_0

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - R_m^{-1} \nabla \times \nabla \times \mathbf{B} \quad \text{with} \quad R_m = \frac{v_0 L}{\eta}$$

as combined parameter

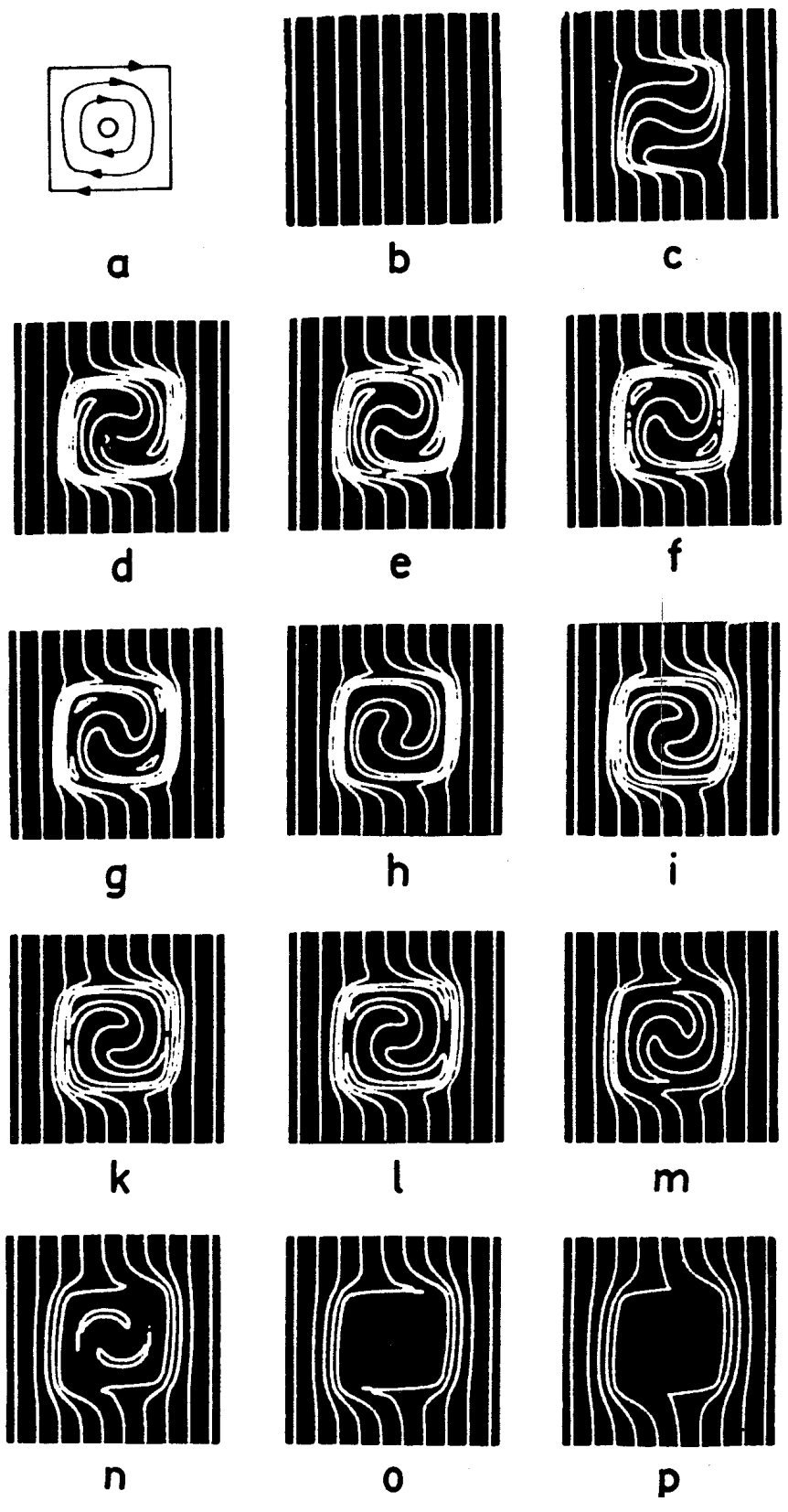
laboratorium: $R_m \ll 1$, cosmos: $R_m \gg 1$

induction for $R_m \gg 1$, diffusion for $R_m \ll 1$, e.g. for small L

flux expulsion from closed velocity fields

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6. MHD equilibria

6.1 Lorentz force

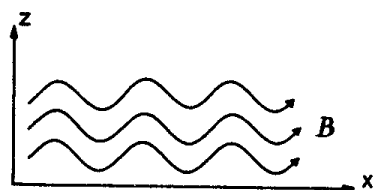
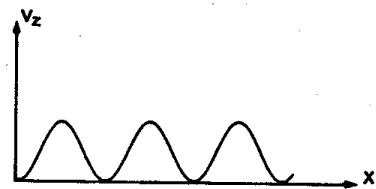
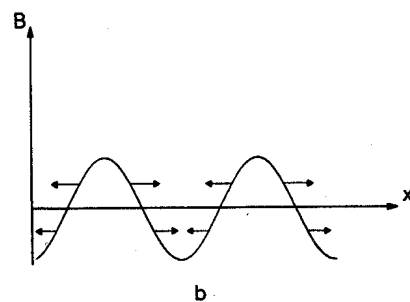
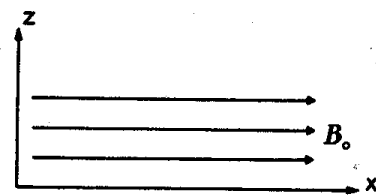
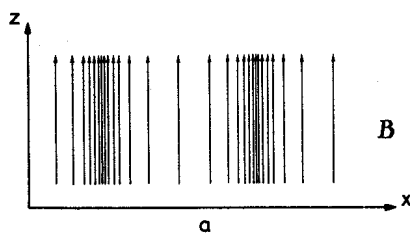
$$\frac{1}{c} \mathbf{j} \times \mathbf{B} = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \frac{B^2}{8\pi} + \frac{1}{4\pi} (\mathbf{B} \cdot \nabla) \mathbf{B}$$

magnetic pressure

$$\mathbf{B} = (0, 0, B(x)), \quad \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{8\pi} \left(\frac{dB^2}{dx}, 0, 0 \right)$$

magnetic tension

$$\mathbf{B} = (B_0, 0, v'_z B_0 dt), \quad \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{4\pi} (0, 0, v''_z B_0^2 dt)$$



6.2 Magnetohydrostatics

$$-\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} (+\rho \mathbf{g}) = 0, \quad \nabla \cdot \mathbf{B} = 0$$

plasma beta $\beta = \frac{\rho}{B^2/8\pi}$

6.2.1 Plasma cylinder

(i) driven by constant current along cylinder

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \theta} = 0, \quad j_z = \text{const} \quad \curvearrowright \quad B_r = B_z = 0$$

$$\frac{\partial}{\partial r} \left(\rho + \frac{B_\theta^2}{8\pi} \right) + \frac{B_\theta^2}{4\pi r} = 0 \quad \text{and} \quad \frac{1}{r} \frac{\partial}{\partial r} (r B_\theta) = \frac{4\pi}{c} j_z = \text{const}$$

$$B_\theta = \frac{2\pi}{c} j_z r \quad \text{inside}, \quad B_\theta = \frac{2\pi}{c} j_z \frac{a^2}{r} \quad \text{outside}, \quad \rho = \rho_0 - \frac{\pi j_z^2 r^2}{c^2} \quad \text{inside}$$

(ii) driven by axial surface current only

$$\rho_0 = \frac{B_0^2}{8\pi} \quad \text{on surface } r = a$$

$$B_\theta = 0 \quad \text{inside}, \quad B_\theta = \frac{C}{r} \quad \text{outside}$$

6.2.2 Force-free fields

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu \mathbf{B}, \quad 0 = \nabla \cdot \mu \mathbf{B} = \mathbf{B} \cdot \nabla \mu$$

μ constant along field lines

(i) $\mu = \text{const}$: cylindrical coordinates, cylind. and axial symmetry

$$-\frac{\partial B_z}{\partial r} = \mu B_\theta, \quad \frac{1}{r} \frac{\partial}{\partial r} r B_\theta = \mu B_z \leadsto B_z = B_0 J_0(\mu r), \quad B_\theta = B_0 J_1(\mu r)$$

(ii) $\mu \neq \text{const}$: magnetic energy density $F(r) = (B_\theta^2 + B_z^2) / 8\pi$

as generating function

$$B_\theta^2 = -4\pi r \frac{dF}{dr}, \quad B_z^2 = 8\pi \left(F + \frac{1}{2} r \frac{dF}{dr} \right) \leadsto \frac{dF}{dr} \leq 0, \quad F + \frac{1}{2} r \frac{dF}{dr} \geq 0$$

exercise : $F = \frac{B_0^2}{8\pi} \frac{1}{a^2 + r^2} \leadsto B_\theta, B_z, \mu$

6.2.3 Magnetohydrostatics of a prominence

cool, dense clouds in solar corona along inversion lines

$$-\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} + \rho \mathbf{g} = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad p = R\rho T$$

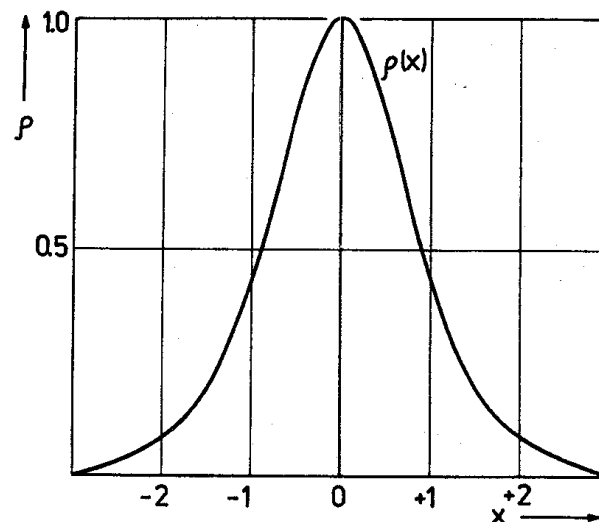
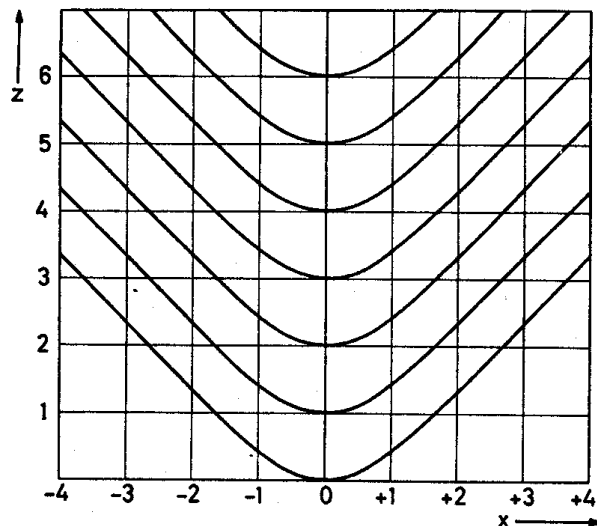
$$\mathbf{B} = (B_x, 0, B_z), \quad \mathbf{g} = (0, 0, -g), \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0, \quad T = \text{const}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \curvearrowright \quad B_x = \text{const}$$

$$RT \frac{\partial \rho}{\partial x} = -\frac{1}{4\pi} B_z \frac{\partial B_z}{\partial x}, \quad 0 = \frac{1}{4\pi} B_x \frac{\partial B_z}{\partial x} - \rho g$$

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\alpha \partial B_z^2}{2 \partial x} \quad \text{with} \quad \alpha^{-1} = \frac{RT}{g} B_x = \text{const}$$

$$B_z = B_z^\infty \tanh\left(\frac{1}{2}\alpha B_z^\infty x\right) \quad \text{and} \quad \rho = \frac{1}{8\pi RT} \left(\frac{B_z^\infty}{\cosh \frac{1}{2}\alpha B_z^\infty x}\right)^2$$



6.2.4 Flux tubes

$$-\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} - \rho g \mathbf{e}_z = 0, \quad \nabla \cdot \mathbf{B} = 0$$

$$\rho = R\rho T, \quad \frac{\partial}{\partial y} = 0, \quad B_y = 0$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{A} = (0, A, 0), \quad B_x = -\frac{\partial A}{\partial z}, \quad B_z = \frac{\partial A}{\partial x}$$

$$\frac{B_x}{B_z} = \frac{dx}{dz}, \quad dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial z} dz = 0, \quad A = \text{const. field lines}$$

$$\frac{\partial p}{\partial x} + \frac{1}{4\pi} \nabla^2 A \frac{\partial A}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} + \frac{1}{4\pi} \nabla^2 A \frac{\partial A}{\partial z} + \rho g = 0$$

$$(x, z) \rightarrow (A, z)$$

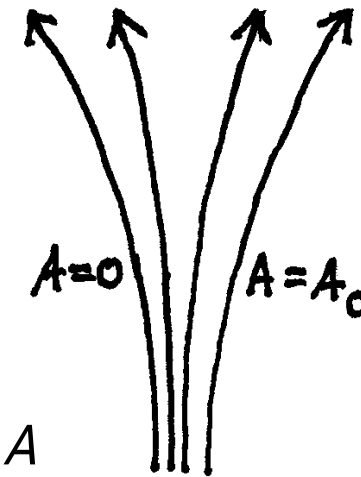
$$\left(\frac{\partial p}{\partial A} \right)_z + \frac{1}{4\pi} \nabla^2 A = 0 \quad \text{and} \quad \left(\frac{\partial p}{\partial z} \right)_A + \rho g = 0$$

2nd eq. hydrostatics along field lines

$$\leadsto \rho(A, z) = \rho_0(A) \exp \left\{ - \int \frac{g}{RT(A, z)} dz \right\}, \quad \rho(A, z)$$

1st eq. Grad – Shafranov equation for $A(x, z)$

$$\text{quite difficult, esp. because of boundaries, } \rho + \frac{B^2}{8\pi} = \rho_e$$



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$$\nabla^2 A = \frac{1}{2} \left(\frac{\partial B^2}{\partial A} \right)_z - \left(\frac{\partial B_x}{\partial z} \right)_A, \quad \frac{\partial}{\partial A} \left(\rho + \frac{B^2}{8\pi} \right)_z = \frac{1}{4\pi} \left(\frac{\partial B_x}{\partial z} \right)_A$$

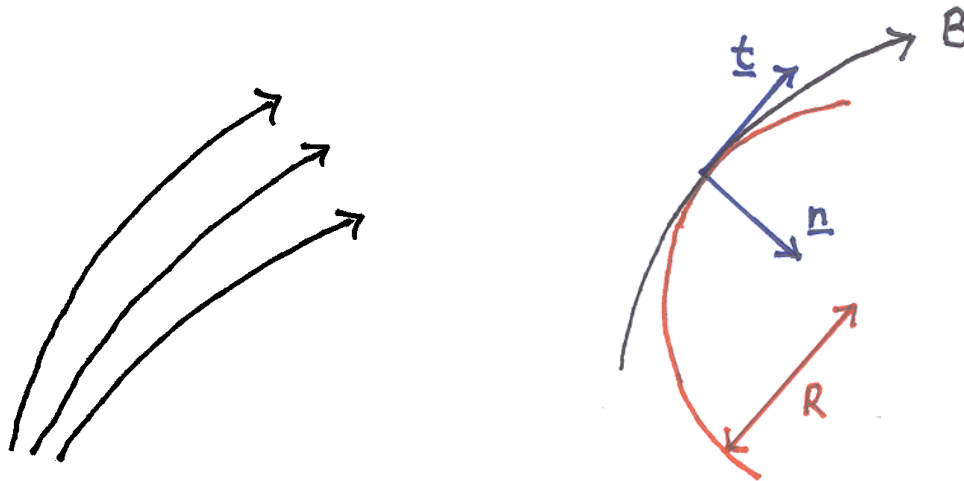
$$\rho + \frac{B^2}{8\pi} = \frac{1}{4\pi} \int \left(\frac{\partial B_x}{\partial z} \right)_A dA + \rho_e(z)$$

$$\int \left(\frac{\partial B_x}{\partial z} \right)_A dA = 0 \quad \text{at boundaries}$$

thin flux tubes ... = 0 everywhere

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6.2.5 Thin flux tubes



$r = r(s)$, s arc – length

$\hat{t} = \frac{\partial r}{\partial s}$, $\hat{n} = R \frac{\partial \hat{t}}{\partial s}$, $\hat{b} = \hat{t} \times \hat{n}$ unit vectors, R curvature radius

Serret – Frenet for $\tau = 0$: $\frac{\partial \hat{n}}{\partial s} = -R \hat{t}$, $\frac{\partial \hat{b}}{\partial s} = 0$, $' = \frac{\partial}{\partial x}$

$$ds = \sqrt{dx^2 + dz^2}, \quad R = \frac{(1+z'^2)^{3/2}}{z''}, \quad \mathbf{e}_z \cdot \hat{t} = \frac{z'}{\sqrt{1+z'^2}}, \quad \mathbf{e}_z \cdot \hat{n} = \frac{1}{\sqrt{1+z'^2}}$$

$\rho + \frac{B^2}{8\pi} = p_e$ lateral pressure balance

$-\frac{\partial p_e}{\partial z} - \rho g = 0$ external stratification

$$-\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} - \rho g \mathbf{e}_z \quad | \cdot \hat{\mathbf{t}}, \cdot \hat{\mathbf{n}}, \cdot \hat{\mathbf{b}}$$

$$-\frac{\partial p}{\partial s} - \rho g(\mathbf{e}_z \cdot \hat{\mathbf{t}}) = 0 \quad \leadsto \quad -\frac{\partial p}{\partial z} - \rho g = 0$$

$$-(\rho - \rho_e)g(\mathbf{e}_z \cdot \hat{\mathbf{n}}) + \frac{B^2}{4\pi R} = 0 \quad \leadsto \quad -(\rho - \rho_e)g + \frac{z''}{1 + z'^2} \frac{B^2}{4\pi} = 0$$

$$-(\rho - \rho_e)g(\mathbf{e}_z \cdot \hat{\mathbf{b}}) = 0 \quad \text{tube in plane } \perp \hat{\mathbf{b}} \text{ for } \rho \neq \rho_e$$

$T(z), T_e(z)$ given

$p(z), p_e(z)$ from tangential component and external stratification

$B^2/8\pi$ from lateral pressure balance

$$2 \frac{z''}{1 + z'^2} = -\frac{g(\rho - \rho_e)}{\rho - \rho_e} = \frac{d/dz(p_e - p)}{\rho_e - \rho} \quad \text{from normal component}$$

$x \rightarrow z$ independent variable, $z' = u$ dependent variable

$$z'' = \frac{du}{dx} = \frac{du dz}{dz du} = u \frac{du}{dz} = \frac{1}{2} \frac{du^2}{z} = \frac{1}{2} \frac{dz'^2}{dz}$$

$$\frac{dz'^2/dz}{1 + z'^2} = \frac{d}{dz} \log(p_e - p), \quad \log(1 + z'^2) = \log(p_e - p)$$

$$z' = \sqrt{C(p_e - p) - 1}, \quad z(x) \text{ by quadrature}$$

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6.3 Magnetohydrodynamics of stellar winds

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \phi} = \frac{\partial}{\partial \theta} = 0, \quad \mathbf{v}(r) = (v_r, 0, v_\phi), \quad \mathbf{B}(r) = (B_r, 0, B_\phi)$$

$$\nabla \cdot (\rho \mathbf{v}) = 0 \quad \leadsto \quad r^2 \rho v_r = \text{const} = I$$

$$\nabla \cdot \mathbf{B} = 0 \quad \leadsto \quad r^2 B_r = \text{const} = \Phi$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \quad \leadsto \quad r(v_\phi B_r - v_r B_\phi) = \text{const} = \Omega \Phi \quad (*)$$

$$\rho(\mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B} - \rho g \mathbf{e}_z$$

$$\phi\text{-component} \quad \leadsto \quad r v_\phi - \frac{\Phi}{4\pi I} r B_\phi = \text{const} = D \quad (**)$$

$$(*) \text{ and } (**) \quad \leadsto \quad r v_\phi = \frac{M_A^2 D - \Omega r^2}{M_A^2 - 1} \quad \text{with}$$

$$M_A^2 = \frac{4\pi I^2}{\rho \Phi^2} = \frac{v_r^2}{B_r^2 / 8\pi} = \left(\frac{v_r}{v_A} \right)^2 \quad \text{Alfven Mach number}$$

$M_A < 1$ at Sun, $M_A > 1$ at Earth

$M_A = 1$ at $r = r_A$ Alfven radius $\sim 10 \dots 20 R_\odot \quad \leadsto \quad D = r_A^2 \Omega$

$$r v_\phi = r_A^2 \Omega \frac{M_A^2 - (r/r_A)^2}{M_A^2 - 1} \quad \rightarrow \quad r_A^2 \Omega \quad \text{for } r \rightarrow \infty$$

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$$\frac{B_r}{B_\phi} = \frac{M_A^2 - 1}{M_A^2(r_A^2 - r^2)\Omega} r v_\phi < 0 \quad \rightarrow \quad 0 \quad \text{for} \quad r \rightarrow \infty$$

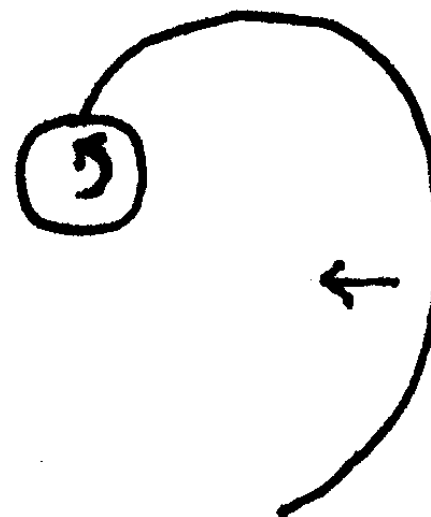
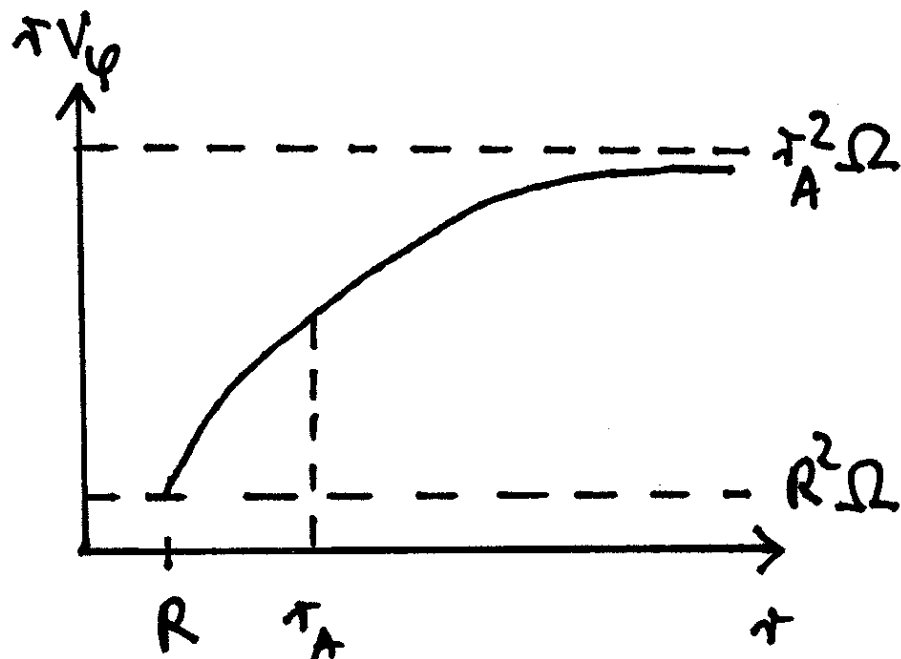
angular momentum per unit mass at ∞ : $r_A^2 \Omega$

radial wind under rigid guidance until r_A

conservation of angular momentum \leadsto magnetic braking

$$(M + \Delta M)R^2\Omega = (MR^2 + \Delta M r_A^2)(\Omega - \Delta\Omega)$$

$$\frac{\Delta\Omega}{\Omega} = \frac{\Delta M}{M} \left(\frac{r_A^2}{R^2} - 1 \right) \quad \text{or} \quad \frac{\dot{\Omega}}{\Omega} = \frac{\dot{M}}{M} \left(\frac{r_A^2}{R^2} - 1 \right)$$


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7. MHD waves

perturbation of a (static) equilibrium \leadsto instability, waves

instability: complicated configurations

now waves: simple equilibrium, e.g. homogenous medium

7.1 Linearisation

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1(\mathbf{x}, t), \quad \mathbf{v} = \mathbf{v}_1(\mathbf{x}, t), \quad p = p_0 + p_1(\mathbf{x}, t), \quad \rho = \rho_0 + \rho_1(\mathbf{x}, t)$$

index 0: equilibrium, static, homogenous, satisfies MHD eqs.

$$\mathbf{B}_0 = \text{const}, \quad \mathbf{j}_0 = 0, \quad \mathbf{v}_0 = 0, \quad p_0 = \text{const}, \quad \rho_0 = \text{const}$$

index 1: perturbations \ll equilibrium

adiabatic perturbations $p_1 = c_s^2 \rho_1$ with $c_s^2 = \gamma p_0 / \rho_0 = \text{const}$

c_s adiabatic sound speed

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0, \quad \frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \frac{1}{4\pi} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0$$

elimination of ρ_1 , ρ_1 and \mathbf{B}_1 in favour of \mathbf{v}_1

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} = c_s^2 \nabla(\nabla \cdot \mathbf{v}_1) + [\nabla \times \{ \nabla \times (\mathbf{v}_1 \times \mathbf{v}_A) \}] \times \mathbf{v}_A$$

$\mathbf{v}_A = \mathbf{B}_0 / \sqrt{4\pi\rho_0}$ Alfvén velocity

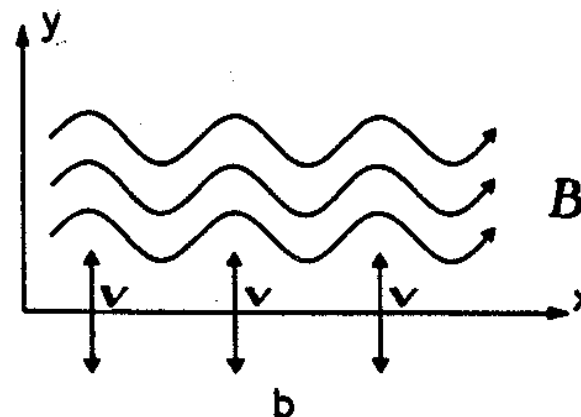
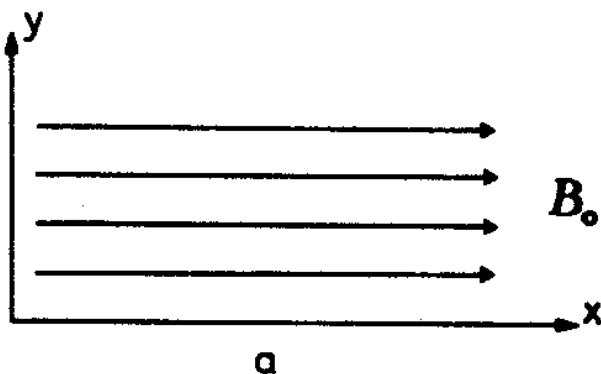
linear, homogenous, constant coefficients, $\mathbf{v}_1 \sim \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$

$$\omega^2 \mathbf{v}_1 = (c_s^2 + v_A^2) (\mathbf{k} \cdot \mathbf{v}_1) \mathbf{k} + \mathbf{v}_A \cdot \mathbf{k} [(\mathbf{v}_A \cdot \mathbf{k}) \mathbf{v}_1 - (\mathbf{v}_A \cdot \mathbf{v}_1) \mathbf{k} - (\mathbf{k} \cdot \mathbf{v}_1) \mathbf{v}_A]$$

7.2 Alfvén waves

$$\mathbf{v}_1 \perp (\mathbf{v}_A, \mathbf{k}) \quad \leadsto \quad \omega = \mathbf{v}_A \cdot \mathbf{k} = \pm v_A k \cos \theta \quad \text{and} \quad \nabla_{\mathbf{k}} \omega = \mathbf{v}_A$$

transversal, dispersion free, magnetic tension


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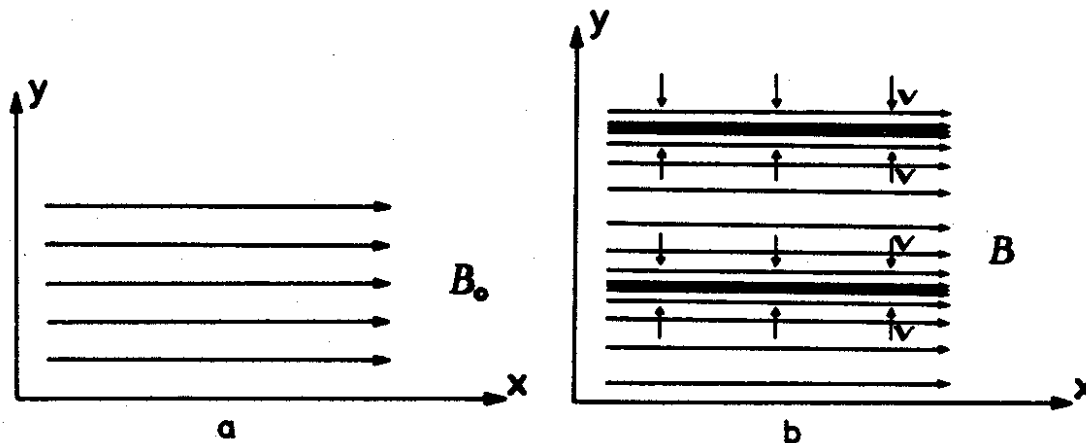
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7.3 Compressional waves

$$k \parallel v_1 \perp v_A \perp k \quad \leadsto \quad \omega/k = \sqrt{c_s^2 + v_A^2}$$

longitudinal, dispersion free, pressure and magnetic pressure gradient



7.4 Waves inclined to magnetic field

$$\angle(\mathbf{v}_A, \mathbf{k}) = \theta$$

dispersion relation vector equation \leadsto three scalar equations

vanishing secular determinant \leadsto three ω^2 for each \mathbf{k}

one solution is the Alfvén wave

the other two are the fast and slow magnetoacoustic waves

$$v_{ph}^2 = \begin{cases} (v_A \cos \theta)^2 \\ \frac{1}{2} (c_s^2 + v_A^2) \pm \frac{1}{2} \left[(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2 \theta \right]^{1/2} \end{cases}$$

pressure and magnetic restoring forces roughly in phase

\leadsto fast waves

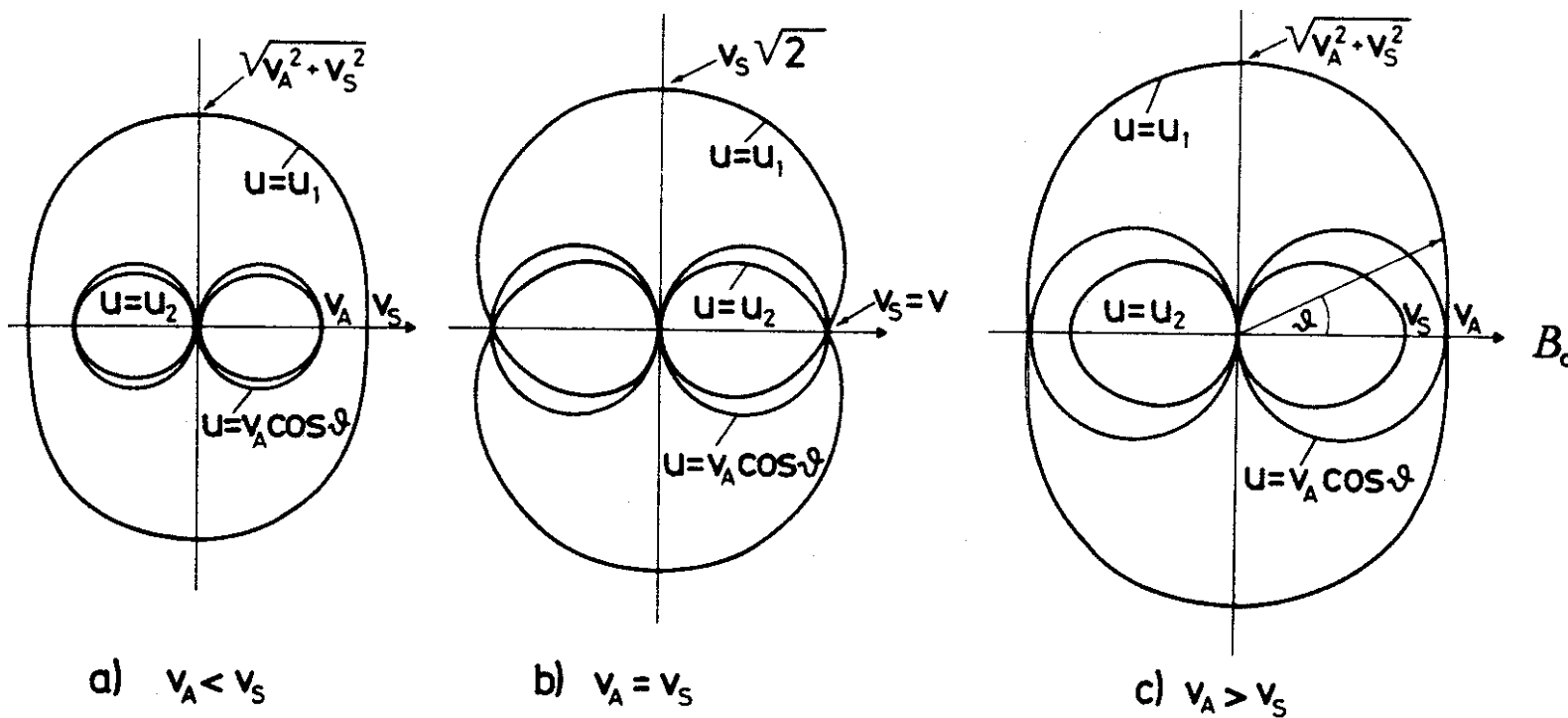
roughly out of phase \leadsto slow waves

displacements of three modes make a triad of orthogonal vectors

\leadsto any disturbance superposition of Alfvén, fast and slow modes

MHD

D. Schmitt



$$u_1 > \max(v_A, v_s), \quad u_2 < \min(v_A, v_s)$$

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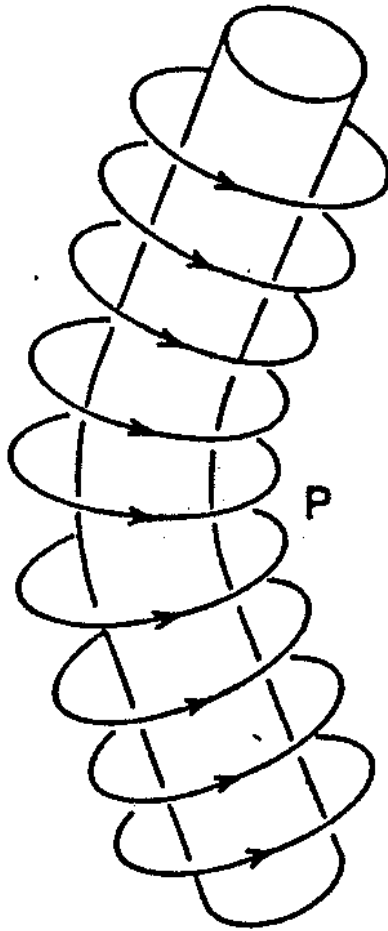
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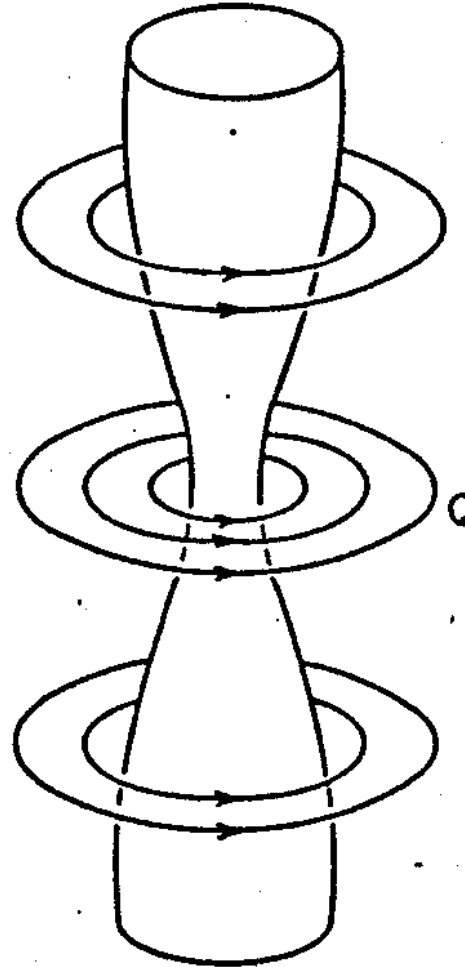
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8. MHD stability

8.1 Plasma cylinder with toroidal field outside



$$B_{\text{left}}^2/8\pi < \rho_0 < B_{\text{right}}^2/8\pi$$



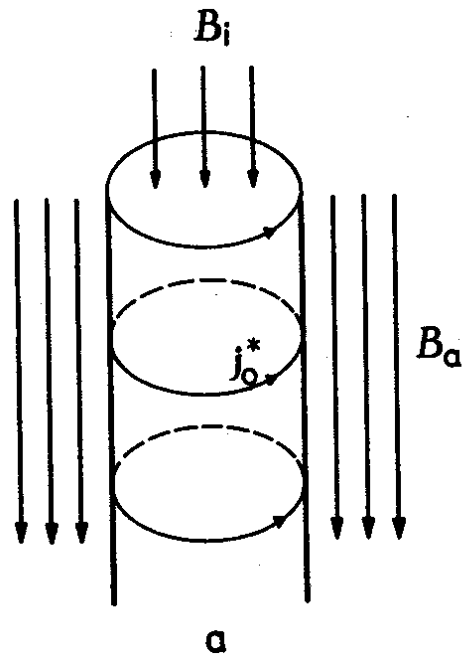
$$\rho_0 < B_{\text{waist}}^2/8\pi$$

8.2 Plasma cylinder with homog. field outside

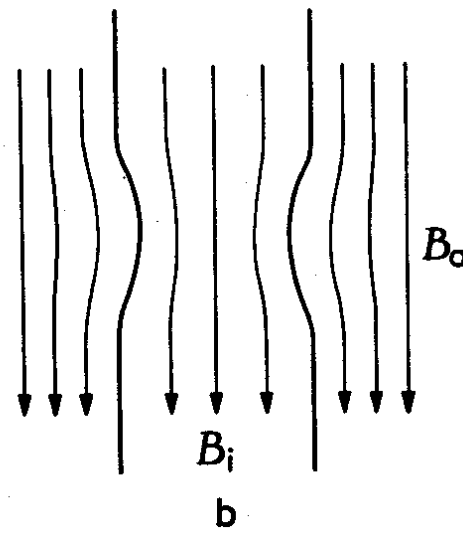
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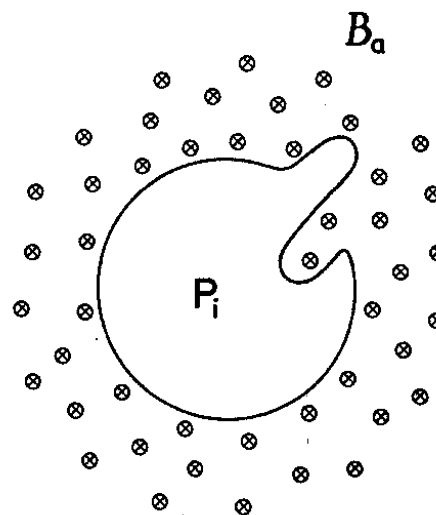
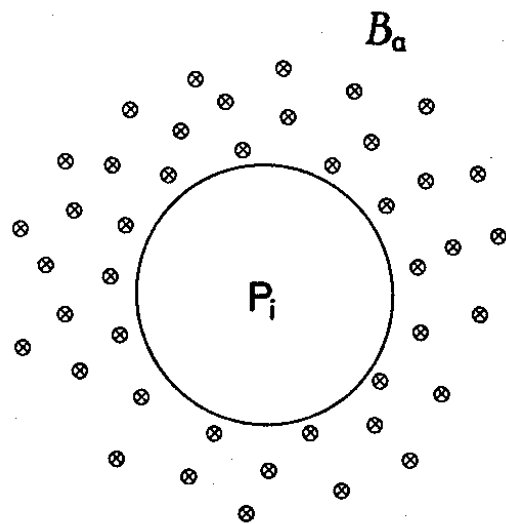
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$$\rho_i + B_i^2/8\pi = B_e^2/8\pi$$



$$B_{e,1}^2 < B_{e,0}^2 \text{ and } B_{i,1}^2 > B_{i,0}^2$$



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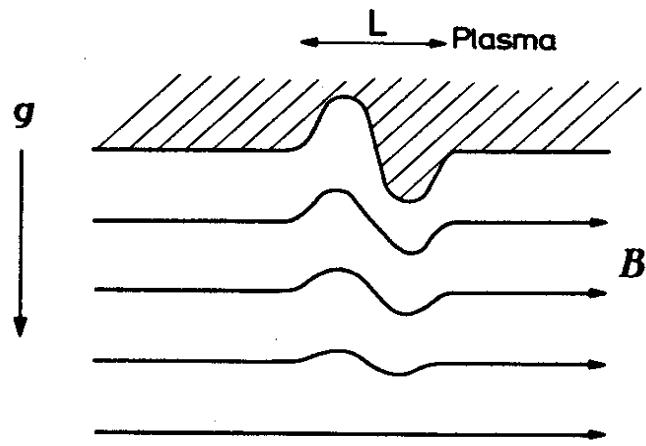
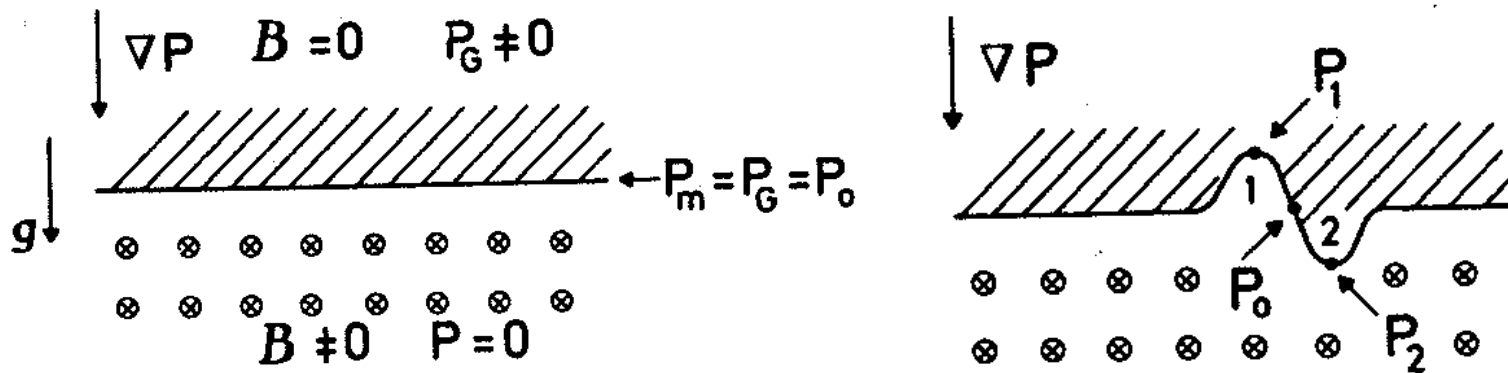
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8.3 Rayleigh-Taylor instability



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8.4 Normal modes analysis

static equilibrium, small adiabatic perturbations

time-dependence of perturbations

formalism similar to waves, but equilibrium spatial-dependent

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = L \mathbf{v}_1 \quad \text{with}$$

$$L \mathbf{v}_1 = \nabla [(\mathbf{v}_1 \cdot \nabla) \rho_0 + \gamma \rho_0 \nabla \cdot \mathbf{v}_1] + \frac{1}{4\pi} (\nabla \times \mathbf{B}_0) \times [\nabla \times (\mathbf{v}_1 \times \mathbf{B}_0)] \\ + \frac{1}{4\pi} \nabla \times \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) \times \mathbf{B}_0 - (\nabla \cdot \rho_0 \mathbf{v}_1) \mathbf{g}$$

real, linear, homogenous, time – independent differential operator

$$\mathbf{v}_1 = \hat{\mathbf{v}}_1(\mathbf{x}) \exp i\omega t \quad \leadsto \quad -\rho_0 \omega^2 \hat{\mathbf{v}}_1 = L \hat{\mathbf{v}}_1$$

$\omega_i = \Im(\omega)$ determines stability

$\omega_i < 0 \leadsto \mathbf{v}_1$ grows exponentially \leadsto instability

boundary conditions: $\mathbf{v}_1 \cdot \mathbf{n} = 0$, $\mathbf{B} \cdot \mathbf{n} = 0$, $\mathbf{E} \times \mathbf{n} = 0$

or infinite system with $\mathbf{v}_1, \mathbf{B}_1, \dots \rightarrow 0$ for $|\mathbf{x}| \rightarrow \infty$

operator L self-adjoint: $\int \mathbf{u} \cdot L\mathbf{v} dV = \int \mathbf{v} \cdot L\mathbf{u} dV$

\leadsto eigenvalues ω_k^2 real, no overstability

eigenfunctions \mathbf{v}_k orthogonal and complete

$(\omega_k^2, \mathbf{v}_k)$ normal modes

$\forall \omega_k^2 \geq 0 \Leftrightarrow$ stability, oscillation or wave

$\exists \omega_k^2 < 0 \Leftrightarrow$ monotonous instability

lower bound to ω^2 : $-\omega^2 = \frac{\int \mathbf{v}_1 \cdot L\mathbf{v}_1 dV}{\int \rho_0 \mathbf{v}_1^2 dV}$ variational form

find \mathbf{v}_1 which maximizes $\int \mathbf{v}_1 \cdot L\mathbf{v}_1 dV$

example: hydrodynamic Rayleigh-Taylor instability

$\frac{d}{dz} \left(\rho_0 \omega^2 \frac{dv_{1z}}{dz} \right) = k^2 \left(\rho_0 \omega^2 - g \frac{d\rho_0}{dz} \right) v_{1z}$ eigenvalue equation

$-\omega^2 = \int \frac{d\rho_0}{dz} g v_{1z}^2 dz \left\{ \int \rho_0 \left[\frac{1}{k^2} \left(\frac{dv_{1z}}{dz} \right)^2 + v_{1z}^2 \right] dz \right\}^{-1}$ incompressible

if $d\rho_0/dz > 0$ somewhere $\leadsto \omega^2 < 0 \leadsto$ instability

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compressible Rayleigh-Taylor instability

$$-\omega^2 = \int \left(\frac{d\rho_0}{dz} + \frac{\rho^2 g}{\gamma \rho_0} \right) g v_{1z}^2 dz \left\{ \int \rho_0 \left[\frac{1}{k^2} \left(\frac{dv_{1z}}{dz} \right)^2 + v_{1z}^2 \right] dz \right\}^{-1}$$

Schwarzschild's convection criterium

magnetic Rayleigh-Taylor instability, incompressible

$$-\omega^2 = \int \left\{ \frac{d\rho_0}{dz} g v_{1z}^2 - (\mathbf{k} \cdot \mathbf{B}_0)^2 \left[\frac{1}{k^2} \left(\frac{dv_{1z}}{dz} \right)^2 + v_{1z}^2 \right] \right\} dz \left\{ \int \rho_0 \left[\frac{1}{k^2} \left(\frac{dv_{1z}}{dz} \right)^2 + v_{1z}^2 \right] dz \right\}^{-1}$$

8.5 Small displacements and energy integral

displacement ξ such that $\mathbf{x} = \mathbf{x}_0 + \xi$, $\mathbf{v}_1(\mathbf{x}) = \partial \xi / \partial t|_{\mathbf{x}_0} = \mathbf{v}_1(\mathbf{x}_0)$

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = L \mathbf{v}_1 \quad \leadsto \quad \rho_0 \frac{\partial^3 \xi}{\partial t^3} = L \frac{\partial \xi}{\partial t} \quad \leadsto \quad \rho_0 \frac{\partial^2 \xi}{\partial t^2} = L \xi$$

multiplication by $\partial \xi / \partial t$ and integration, L self-adjoint:

$$\frac{\partial}{\partial t} \frac{1}{2} \int \rho_0 \left(\frac{\partial \xi}{\partial t} \right)^2 dV = \int \rho_0 \frac{\partial \xi}{\partial t} \cdot \frac{\partial^2 \xi}{\partial t^2} dV = \int \frac{\partial \xi}{\partial t} \cdot L \xi dV = \frac{\partial}{\partial t} \frac{1}{2} \int \xi \cdot L \xi dV$$

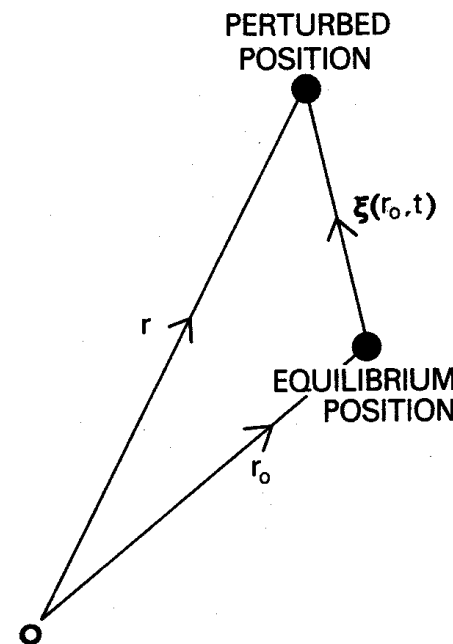
$$\frac{1}{2} \int \rho_0 \left(\frac{\partial \xi}{\partial t} \right)^2 dV = \frac{1}{2} \int \xi \cdot L \xi dV, \quad \delta K = -\delta W$$

$\forall \xi : \delta W > 0 \leadsto$ stability

$\exists \xi : \delta W < 0 \leadsto$ instability



minimization of δW with respect to all admissible ξ



8.6 Energy method

$$\delta W = -\frac{1}{2} \int \xi \cdot \left\{ \nabla [\xi \cdot \nabla \rho_0 + \gamma \rho_0 \nabla \cdot \xi] + \frac{1}{4\pi} (\nabla \times B_0) \times [\nabla \times (\xi \times B_0)] \right. \\ \left. + \frac{1}{4\pi} \nabla \times \nabla \times (\xi \times B_0) \times B_0 - (\nabla \cdot \rho_0 \xi) \mathbf{g} \right\} dV$$

$$\mathbf{a} \cdot \nabla \lambda = \nabla \cdot (\lambda \mathbf{a}) - \lambda \nabla \cdot \mathbf{a}, \quad \mathbf{a} \cdot (\nabla \times \mathbf{b}) = -\nabla \cdot (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \cdot (\nabla \times \mathbf{a})$$

$$\int \nabla \cdot \mathbf{a} dV = \int \mathbf{a} \cdot d\mathbf{F}$$

$$\delta W = \frac{1}{2} \int \left\{ [\xi \cdot \nabla \rho_0 + \gamma \rho_0 \nabla \cdot \xi] \nabla \cdot \xi - \frac{1}{4\pi} \xi \cdot [(\nabla \times B_0) \times (\nabla \times (\xi \times B_0))] \right. \\ \left. + \frac{1}{4\pi} [\nabla \times (\xi \times B_0)]^2 + (\nabla \cdot \rho_0 \xi) \mathbf{g} \right\} dV \\ - \frac{1}{2} \int \left\{ \xi [\xi \cdot \nabla \rho_0 + \gamma \rho_0 \nabla \cdot \xi] + \frac{1}{4\pi} [(\xi \times B_0) \times (\nabla \times (\xi \times B_0))] \right\} \cdot d\mathbf{F}$$

2nd and 4th term of volume integral positive definite \leadsto stabilizing
acoustic, Alfvén and fast magnetoacoustic waves

1st, 3rd and 5th term: either positive or negative, if negative
potentially hydrodynamic and current-driven instabilities

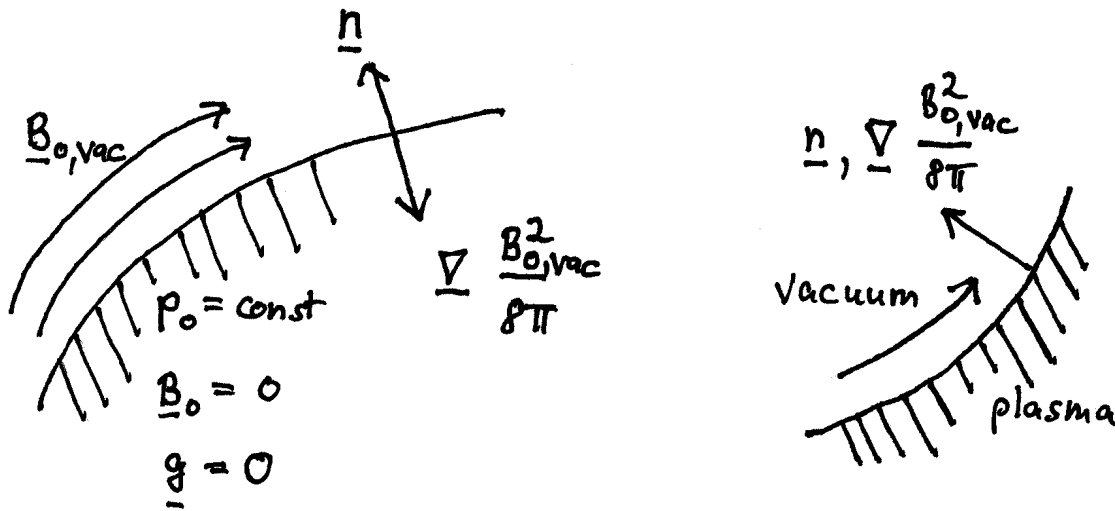
surface integral vanishes for rigid boundaries $\xi = 0$

surface integral on plasma-vacuum boundary:

$$-\frac{1}{2} \int \dots dF = -\frac{1}{2} \int (\xi \cdot n)^2 n \cdot \nabla \left(p_0 + \frac{B_0^2}{8\pi} - \frac{B_{0,vac}^2}{8\pi} \right) dF$$

$$+ \int_{vac} \frac{1}{8\pi} [\nabla \times (\xi \times B_0)]^2 dV$$

stability determined by sign of $n \cdot \nabla(\dots)$, independent of ξ



$$\delta W = \underbrace{\frac{1}{2} \int \gamma p_0 (\nabla \cdot \xi)^2 dV}_{> 0} + \underbrace{\frac{1}{2} \int (\xi \cdot n)^2 n \cdot \nabla \frac{B_{0,vac}^2}{8\pi} dF}_{\geq 0} + \underbrace{\int \frac{1}{8\pi} [\nabla \times (\xi \times B_0)]^2 dV}_{> 0}$$

concave confinement destabilizing, convex confinement stabilizing
fluting instability

8.7 Stationary equilibrium

$$-\rho_0 \omega^2 \xi + 2\omega i A \xi + B \xi = 0$$

$$-\omega^2 \int \rho_0 \xi^2 dV + 2\omega \int i \xi \cdot A \xi dV + \int \xi \cdot B \xi dV = 0$$

$$-\omega^2 a + 2\omega b + c = 0$$

iA Hermitian, B self-adjoint, integrals a , b , c real

stability if $c > 0$ or $b^2 + ac > 0$

potentially overstability

