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Solar System School

FUNDAMENTALS OF MAGNETOHYDRODYNAMICS

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1. **Introduction: the magnetohydrodynamic approximation (MHD).**
   1.1. Maxwell equations. Lorentz transformations.
   1.2. Physical assumptions underlying the MHD approximation.
   1.3. *Ohm's law* as a constitutive relation.
2. **Kinematic preliminaries.**
   2.1. The continuity equation in the spatial and in the material representations. The Jacobian determinant; Euler's identity.
   2.2. Material curves and material regions. Reynold's transport theorem for line and for volume integrals.
   2.3. Magnetic flux tubes.
3. **The magnetic induction equation.**
   3.1. Derivation of the induction equation of MHD.
   3.2. Scale analysis: the magnetic Reynolds number.
   3.3. The limit of infinite conductivity: Alfén's theorem on the conservation of magnetic flux. Waléns theorem on the 'freezing' of magnetic field lines.
   3.4. Magnetic helicity and its topological significance. Gauss' linking number. Conservation of magnetic helicity.
4. **The momentum equation.**
   4.1. Derivation of the equation of motion from the integral expression of momentum balance.
   4.2. The Lorentz force. Magnetic pressure, magnetic tension and curvature force. Maxwell's stress tensor.
   4.3. The theorem for the balance of mechanical energy. Mechanical power; the stress tensor and the deformation power.
5. **The energy equation.**
   5.1. Poynting's theorem for the balance of electromagnetic energy. Poynting's vector and its meaning.
   5.2. The principle of energy conservation. The heat-flux-density vector; heating power. Statement of the 'First principle of Thermodynamics' for a MHD plasma. Ohmic dissipation in resistive MHD.

5.3. The energy equation expressed in terms of entropy. Viscous dissipation, Ohmic dissipation and heat conduction as entropy sources. Isentropic motions.

5.4. Alternative forms of writing the energy equation in MHD in different thermodynamic representations.

6. Magnetostatics.

6.1. Equation of magnetostatic equilibrium. Scale analysis of the different terms.

6.2. Force-free magnetic fields. Theorems.

7. Alfvén waves.

7.1. Alfvén waves in an ideal plasma. Obtention of the wave equation. The Alfvén speed. Polarization properties. Equipartition of energy.

7.2. Alfvén waves in a plasma with electrical resistivity. Damped oscillations.

8. Jump relations across a MHD shock transition.

8.1. The basic equations of MHD in conservation form.

8.2. The basic equations in a frame of reference comoving with the shock discontinuity. Continuity of the normal component for a solenoidal vector or tensor quantity.

8.3. Jump relations for mass conservation, momentum balance and energy conservation. Jump relations for the magnetic and the electric fields.

8.4. Jump condition for the entropy flux density across the shock discontinuity. Implications of this inequality on the direction of change of the other physical quantities across the shock.

8.5. Classification of MHD shocks.

9. Elementary introduction to stellar dynamo theory.

9.1. Formal posing of the dynamo problem.

9.2. Averages and deviations. Derivation of the kinematic dynamo equations.

9.3. Differential rotation and Ω-effect. Mean induced electric field and α-effect. Enhanced magnetic diffusivity.

9.4. The concept of $\alpha\Omega$, α^2 and $\alpha^2\Omega$ dynamos.

9.5. A simple *constitutive relation* introducing the scalar functions α (α-effect) and β (*turbulent diffusivity)*. Parker's original heuristic model for a simple $\alpha\Omega$-dynamo; dynamo waves.

Basic literature for the course MAGNETOHYDRODYNAMICS

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Bibliography on MHD

- **Plasma Dynamics**, T. J. M. **B**oyd & J. J. **S**anderson, Barnes & Noble, New York (1969).
- **Hydrodynamic and Hydromagnetic Stability**, S. **C**handrasekhar, Dover (1981).
- **Magnetohydrodynamics**, T. G. **C**owling, Adam Hilger, Bristol (1976).
- **Electrodynamics of Continuous Media**, L. D. **L**andau, E. M. **L**ifshitz & L. P. **P**itaevskiĭ, vol. **8** of the *Course of Theoretical Physics*, Butterworth-Heinemann, 2 edition (1984).
- **Solar Magnetohydrodynamics**, E. R. **P**riest, Reidel, Dordrecht (1982).
- **Principles of Magnetoplasma Physics**, L. C. **W**oods, Clarendon Press, Oxford, (1987).

The excellent book **Solar Magnetohydrodynamics** is especially recommended for the topics 'Magnetohydrostatics. Force-free fields' (chapter 3), 'MHD waves' (chapter 4) and 'MHD shock waves' (chapter 5).

Complementary bibliography on Hydrodynamics

- **An introduction to fluid mechanics**, G. K. **B**atchelor, Cambridge University Press (1970).
- **The Physics of Fluids and Plasmas: An Introduction for Astrophysicists**, Arnab R. **C**houdhuri, Cambridge University Press (1998).
- **Supersonic flow and shock waves**, **C**ourant & **F**riedrichs, Springer-Verlag (1972).
- **Fluid Mechanics**, L. D. **L**andau & E. M. **L**ifshitz, vol. 6 of the **Course of Theoretical Physics**, Butterworth-Heinemann; 2 edition (1987).
- **Physical fluid dynamics**, D. J. **T**ritton, Oxford Science Publications (1988).
- **Theoretical fluid dynamics**, B. K. **S**hivamoggi, series *Mechanics of fluids and transport processes*, Martinus Nijhoff Publishers, Dordrecht (1985).

The book **Fluid Mechanics** by Landau & Lifshitz contains very clear and extensive chapters on 'Sound Waves' and 'Shock Waves'.

Complementary bibliography on (rational) Thermodynamics

- **Rational Thermodynamics**, Clifford A. **T**ruesdell, Springer-Verlag (1984).
- **The non-linear field theories of mechanics**, Clifford A. **T**ruesdell & Walter **N**oll, Springer-Verlag (1992).
- **The Tragicomical History of Thermodynamics**, Clifford A. **T**ruesdell, Springer-Verlag (1980).

MAGNETOHYDRODYNAMICS

Bulletin of exercises n°1: Kinematic aspects in Continuum Mechanics. The Jacobian and its geometrical interpretation. Equation of continuity.

The motion of a fluid is given by a function

$$\mathbf{x} = \mathbf{X}(\mathbf{a}, t)\,, \tag{1}$$

where $\mathbf{a}$ is the position vector at $t = 0$ of the fluid element which is at position $\mathbf{x}$ at time t: $\mathbf{X}(\mathbf{a}, 0) = \mathbf{a}$.

For each fixed t, (1) defines an invertible transformation of the continuum onto itself. The Jacobian determinant

$$J\,(\mathbf{a}, t) \stackrel{\text{def}}{=} \det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{a}}\right)_t = \det D\mathbf{X}(\mathbf{a}, t)$$

is, therefore, always different from zero.

1. Euler's identity.

For the proof of Reynolds' transport theorem use is made of the so-called *Euler's identity*,

$$\left(\frac{\partial J}{\partial t}\right)_{\mathbf{a}} = J \operatorname{div} \mathbf{v}\,,$$

where J is the Jacobian determinant of the transformation $\mathbf{x} = \mathbf{X}(\mathbf{a}, t)$ and $\mathbf{v}$ is the velocity field.

Prove Euler's identity.

2. Continuity equation in the material representation.

By means of Euler's identity, obtain the equation of continuity in the material (or *lagrangian*) representation, viz.

$$\rho(\mathbf{a}, t)\, J(\mathbf{a}, t) = \rho(\mathbf{a}, 0)\,.$$

3. Incompressible flows.

A flow is said to be incompressible if the volume of any arbitrary portion of the fluid remains constant in time; i.e., if for any arbitrary Ω_{t} it holds that

$$\frac{d}{dt}\int_{\Omega_{\mathrm{t}}} 1 \;=\; 0\,.$$

- Prove that a flow is incompressible **if and only if** the Jacobian J is equal 1 at all times.

- Prove that a flow is incompressible **if and only if** $D\rho/Dt \equiv 0$.

MAGNETOHYDRODYNAMICS

Bulletin of exercises n°2: Kinematic aspects in Continuum Mechanics. Transport theorem for a material region and for a material circuit.

1. Reynolds' transport theorem.

Reynolds' theorem in Continuum Mechanics is basically a re-statement of the theorem of change of variable in a volume integral.

Let Ω_{t} be a "material region" (i.e., a region in 3-D space occupied at time t by a portion of continuum). We require from Ω_{t} to be an open, connected region. (For some applications we may also require that the boundary $\partial\,\Omega_{\mathrm{t}}$ be a piecewise regular surface).

Let $F(\mathbf{x},t)$ be a function of position and time defined for $\mathbf{x}\in\Omega_{\mathrm{t}}$ and $t\in(t_1,t_2)$.

In Hydrodynamics/MHD one often comes across time derivatives of the form

$$\frac{d}{d\,t}\int\limits_{\Omega_{\mathrm{t}}} F(\mathbf{x},t)\ ,$$

where not only the integrand depends on time but also the region of integration.

1.1. By using *Euler's identity* (see bulletin 1) show that

$$\frac{d}{d\,t}\int\limits_{\Omega_{\mathrm{t}}} F(\mathbf{x},t) \;=\; \int\limits_{\Omega_{\mathrm{t}}}\left(\frac{D\,F}{D\,t}+F\,\mathrm{div}\,\mathbf{v}\right) \;=\; \int\limits_{\Omega_{\mathrm{t}}}\left\{\left.\frac{\partial F}{\partial\,t}\right|_{\mathbf{x}}+\mathrm{div}\,(F\,\mathbf{v})\right\}\ , \tag{1}$$

where $D\,/D\,t$ is the material derivative and $\mathbf{v}(\mathbf{x},t)$ is the velocity of each $\mathbf{x}\in\Omega_{\mathrm{t}}$.

1.2. Combining Reynolds' theorem (2.1) with the equation of continuity, prove the following result:

$$\frac{d}{d\,t}\int\limits_{\Omega_{\mathrm{t}}} \rho\,F(\mathbf{x},t) \;=\; \int\limits_{\Omega_{\mathrm{t}}} \rho\,\frac{D\,F}{D\,t}\ . \tag{2}$$

The above result is usually called *Reynolds' transport theorem*; it is no a longer a purely mathematical result, since use has been made of the continuity equation, which expresses mass conservation.

2. Transport theorem for a material curve:

Let $\boldsymbol{\alpha}=\boldsymbol{\alpha}(\xi,t)$ be the parametric expression of the material contour or circuit C_{t} (a simple, closed curve made up at all times of the same material elements). Let $\mathbf{Q}(\mathbf{x},t)$ be a vector quantity defined in the flow region. The circulation of $\mathbf{Q}$ around the circuit C_{t} is defined as

$$\Gamma(C_{\mathrm{t}}) = \oint_{C_{\mathrm{t}}}\mathbf{Q}\cdot\mathrm{d}\boldsymbol{\alpha} \stackrel{\mathrm{def}}{=} \int_{\xi_1}^{\xi_2}\mathbf{Q}[\boldsymbol{\alpha}(\xi,\mathrm{t})]\cdot\boldsymbol{\alpha}'(\xi,\mathrm{t})\,\mathrm{d}\xi\,, \tag{3}$$

where $\xi\in(\xi_1,\xi_2)$ and $\boldsymbol{\alpha}'(\xi,t)$ is the short-hand notation for the tangent vector at (ξ,t). Prove that

$$\frac{d}{dt}\oint_{C_{\mathrm{t}}}\mathbf{Q}\cdot d\boldsymbol{\alpha} = \oint_{C_{\mathrm{t}}}\left[\frac{\partial\mathbf{Q}}{\partial t}+(\mathbf{rot}\,\mathbf{Q})\wedge\mathbf{v}\right]\cdot d\boldsymbol{\alpha}\,. \tag{4}$$

MAGNETOHYDRODYNAMICS

Bulletin of exercises n°3: Some results in Electromagnetism.

1. Continuity equation for the electric charge.

- Starting from Maxwell's equations obtain a 'continuity equation' for magnetic charge (i.e., a differential equation expressing the conservation of magnetic charge).

- Interpret the continuity equation for the electric charge by integrating it in a fixed region $\mathcal{R}$ with boundary $\partial\mathcal{R}$. Draw a parallelism with the continuity equation of Hydrodynamics (i.e., the differential equation expressing the conservation of mass).

2. Poynting's vector and Poynting's theorem.

If we have a continuous distribution of charges and currents, the total work per unit time (i.e., *power*) exerted by the electromagnetic fields on the matter in a fixed region $\mathcal{R}$ in space is:

$$\mathcal{W}^{\mathrm{em}} = \int_{\mathcal{R}} \mathbf{j}\cdot\mathbf{E}\,. \tag{1}$$

- Starting from Maxwell's equations (Faraday's and Ampere's laws), obtain a differential equation relating this power $\mathcal{W}^{\mathrm{em}}$ with the rate of change of the total electromagnetic energy contained in $\mathcal{R}$ and with the energy flux through the boundary $\partial\mathcal{R}$.

- Introducing the following definitions,

$$\begin{cases} \mathcal{E}_{\mathrm{em}} = \dfrac{\|\mathbf{B}\|^2}{8\pi} + \dfrac{\|\mathbf{E}\|^2}{8\pi} & \text{[electromagnetic energy density]} \\ \mathbf{S} = \dfrac{c}{4\pi}\,\mathbf{E}\wedge\mathbf{B}\,, & \text{[Poynting's vector]} \end{cases} \tag{2}$$

write the differential equation in the form

$$\frac{\partial\mathcal{E}_{\mathrm{em}}}{\partial t} + \operatorname{div}\mathbf{S} = -\mathbf{j}\cdot\mathbf{E}\,. \tag{3}$$

- Interpret Eq. (3) [Poynting's theorem] by integrating it in a fixed region $\mathcal{R}$ with boundary $\partial\mathcal{R}$. Draw a parallelism with the continuity equation of Hydrodynamics in the case of chemical reactions.

3. Poynting's theorem for a material region Ω_{t}.

Poynting's theorem [eq. (3)] has been derived for a fixed region $\mathcal{R}$. Now we extend it to the case of a material region Ω_{t} by using Reynolds' theorem.

Show that in the case of a material region Ω_{t}, the integral expression of Poynting's theorem is

$$\dot{E}_{\mathrm{em}}(\Omega_{\mathrm{t}}) = \frac{d}{dt}\int_{\Omega_{\mathrm{t}}} \mathcal{E}_{\mathrm{em}} = -\int_{\Omega_{\mathrm{t}}} \mathbf{j}\cdot\mathbf{E} - \oiint_{\partial\Omega_{\mathrm{t}}} \mathbf{S}\cdot\mathbf{n} + \oiint_{\partial\Omega_{\mathrm{t}}} \mathcal{E}_{\mathrm{em}}\,\mathbf{v}\cdot\mathbf{n}\,. \tag{4}$$

where $E_{\mathrm{em}}(\Omega_{\mathrm{t}})$ is the total electromagnetic energy contained in the material region Ω_{t}, $\mathcal{E}_{\mathrm{em}}$ is the corresponding density and $\mathbf{v}(\mathbf{x},t)$ is the velocity at each point $\mathbf{x}\in\partial\Omega_{\mathrm{t}}$ of the boundary.

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Bulletin of exercises n°4: The magnetic induction equation.

1. The induction equation: Starting from Faraday's law and Ampere's law (neglecting the displacement current) and making use of Ohm's constitutive relation, eliminate the electric field and the current density to obtain the inducion equation for a plasma in the MHD approximation:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{rot}\,(\mathbf{v} \wedge \mathbf{B}) - \mathbf{rot}\left(\frac{c^2}{4\pi\sigma_{\mathrm{e}}}\,\mathbf{rot}\,\mathbf{B}\right). \tag{1}$$

Show that if the electrical conductivity is uniform, the induction equation can be cast into the form

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{rot}\,(\mathbf{v} \wedge \mathbf{B}) + \eta\,\nabla^2\,\mathbf{B}\,, \tag{2}$$

where $\eta \stackrel{\text{def}}{=} c^2/(4\pi\sigma_{\mathrm{e}})$ is the "magnetic diffusivity."

2. Combined form of the induction and the continuity equations.

Show that the induction equation can be combined with the equation of continuity into one single differential equation which gives the time evolution of $\mathbf{B}/\rho$ following a fluid element, viz:

$$\frac{D}{D\,t}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)\mathbf{v} \;-\; \frac{1}{\rho}\,\mathbf{rot}\,(\eta\,\mathbf{rot}\,\mathbf{B})\,. \tag{3}$$

In the case of an ideal MHD-plasma, Eq. (3) simplifies to a form known as *Walén's equation*.

3. Strength of a magnetic flux tube.

A *magnetic flux tube* is the region enclosed by the surface determined by all magnetic field lines passing through a given material circuit C_{t} .

The "strength" of a flux tube is defined as the circulation of the potential vector $\mathbf{A}$ along the material circuit C_{t} , viz.

$$\Phi_{\mathrm{m}}(t) \stackrel{\text{def}}{=} \oint_{C_{\mathrm{t}}} \mathbf{A}\cdot d\boldsymbol{\alpha} \stackrel{\text{def}}{=} \int_{\xi_1}^{\xi_2} \mathbf{A}[\boldsymbol{\alpha}(\xi,t)]\cdot\boldsymbol{\alpha}'(\xi,t)\,d\,\xi\,, \tag{4}$$

where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\xi, t)$ is the parametric expression of the circuit C_{t} .

- It is immediate to show that the circulation of $\mathbf{A}$ around the circuit C_{t} must be equal to the flux of $\mathbf{B}$ through any surface Σ_{t} spanned by the the contour C_{t}:

$$\Phi_{\mathrm{m}}(t) = \iint_{\Sigma_{\mathrm{t}}} \mathbf{B}\cdot\mathbf{n}\,. \tag{5}$$

- Prove the following result (which is purely kinematic): *At any instant t, the magnetic flux through any section of a magnetic tube tube is the same.*

- Comment on the fact that the concept of 'tube' is introduced only for solenoidal fields.

4. Walén's theorem on the 'freezing' of magnetic field lines.

THEOREM: Assume a perfectly conducting MHD-plasma. Then, if a material curve is a line of force at an initial time t_0, it will be a magnetic line of force at any later time t.

The above result is expressed in a pictorial way by saying that *the magnetic lines of force are frozen in a perfectly conducting plasma.*

Hints: (a) First pose the problem from a purely geometrical point of view (which condition must be satisfied by a material curve that is initially a magnetic field line in order to keep being a magnetic field line at any later time?). (b) Integrate Walén's equation [i.e., Eq. (3) for the case $\eta = 0$] in the Lagrangian representation. (c) Make use of the identity

$$\frac{\partial X_i}{\partial a_k}\frac{\partial A_k}{\partial x_j} = \delta_{ij}\,, \qquad \text{where} \quad \mathbf{x} = \mathbf{X}\,(\mathbf{a}, t) \quad \text{and} \quad \mathbf{A} \stackrel{\text{def}}{=} \mathbf{X}^{-1}\,. \tag{6}$$

5. Conservation of magnetic helicity.

The quantity $\mathbf{A}\cdot\mathbf{B}$ is called the *magnetic helicity density.* The magnetic helicity of a material region Ω_{t} is defined as

$$\mathcal{H}(\Omega_{\mathrm{t}}) = \int_{\Omega_{\mathrm{t}}} \mathbf{A}\cdot\mathbf{B} \tag{7}$$

and it can be shown to be a *topological quantity* expressing the 'degree of complexity' of the magnetic field lines in the region Ω_{t}. Under some circunstances, the magnetic helicity $\mathcal{H}(\Omega_{\mathrm{t}})$ is a constant of motion.

- Starting from the induction equation obtain the following evolution equation for $\mathbf{A}\cdot\mathbf{B}/\rho$ in the limit of a perfectly conducting MHD-plasma:

$$\frac{D}{D\,t}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho}\cdot\nabla\right)(\mathbf{A}\cdot\mathbf{v} - \phi)\;, \tag{8}$$

where $\phi(\mathbf{x}, t)$ is a differentiable scalar function.

- Integrating Eq. (8) in a material region Ω_{t} such that (permanently) $\mathbf{n}\cdot\mathbf{B} = 0\ \forall\mathbf{x}\in\partial\Omega_{\mathrm{t}}$ show that the magnetic helicity $\mathcal{H}(\Omega_{\mathrm{t}})$ is a conserved quantity, viz. $\dot{\mathcal{H}}(\Omega_{\mathrm{t}}) = 0\,$.

- Deduce from the above that the magnetic helicity of a flux tube in ideal MHD is a constant of motion.

Bulletin of exercises n°5: Energy balance in MHD. The energy equation.

1. The kinetic energy equation:

By scalar multiplication of the momentum equation (i.e., Navier-Stokes equation including Lorentz' force) with the velocity field and after integrating in a material region $\Omega_{\rm t}$ we obtain an integral balance equation for the kinetic energy of the plasma contained in $\Omega_{\rm t}$.

Show that the time variation of the total kinetic energy contained in the material region, viz. $\dot{\mathcal{K}}(\Omega_{\rm t})$, is given by

$$\frac{d}{dt}\int_{\Omega_{\rm t}}\frac{1}{2}\rho\,\|\mathbf{v}\|^2 + \int_{\Omega_{\rm t}}\hat{\boldsymbol{\sigma}}:\hat{\boldsymbol{D}} = \int_{\Omega_{\rm t}}\rho\,\mathbf{g}\cdot\mathbf{v} + \oiint_{\partial\Omega_{\rm t}}\mathbf{v}\cdot\hat{\boldsymbol{\sigma}}\mathbf{n} + \int_{\Omega_{\rm t}}\mathbf{v}\cdot\frac{\mathbf{j}\wedge\mathbf{B}}{c} \tag{1.a}$$

or

$$\dot{\mathcal{K}}\,(\Omega_{\rm t}) + \mathcal{W}^{\,\rm def}\,(\Omega_{\rm t}) = \mathcal{W}^{\,\rm vol}\,(\Omega_{\rm t}) + \mathcal{W}^{\,\rm sur}\,(\partial\Omega_{\rm t}) + \int\mathbf{v}\cdot\frac{\mathbf{j}\wedge\mathbf{B}}{c} \tag{1.b}$$

Here $\hat{\boldsymbol{\sigma}}$ is the stress tensor and $\hat{\boldsymbol{D}}$ is the deformation tensor (i.e., the symmetric part of the gradient-of-velocity tensor). Further, $\mathcal{W}^{\,\rm def}\,(\Omega_{\rm t})$ is the deformation power, $\mathcal{W}^{\,\rm vol}\,(\Omega_{\rm t})$ is the power exerted by the long-range forces –volume forces– on the plasma inside $\Omega_{\rm t}$ and $\mathcal{W}^{\,\rm sur}\,(\partial\Omega_{\rm t})$ is the power exerted by the contact forces –surface forces– on the system through its boundary $\partial\Omega_{\rm t}$.

- What is the physical meaning of the integral $\int_{\Omega_{\rm t}}\mathbf{v}\cdot(\mathbf{j}\wedge\mathbf{B})/c$?
- Show that under some restriction (which one?), the power exerted by the long-range forces on the plasma contained in $\Omega_{\rm t}$ can be expressed as minus the rate of change of the gravitational potential energy, viz. $\mathcal{W}^{\,\rm vol}\,(\Omega_{\rm t}) = -\dot{V}\,(\Omega_{\rm t})$.
- Separate in the integral $\mathcal{W}^{\,\rm def}\,(\Omega_{\rm t}) = \int_{\Omega_{\rm t}}\hat{\boldsymbol{\sigma}}:\hat{\boldsymbol{D}}$ the 'net deformation power' (i.e., the 'useful power') from the power that irreversibly goes into thermal energy through viscous effects.

2. Balance for the kinetic and (electro)magnetic energy budget of a MHD-plasma.

Combining the integral expression of Poynting's theorem for a material region $\Omega_{\rm t}$ [Eq. (4) in bulletin 3] with the kinetic energy theorem derived in exercise 1 [Eq. (1) in this bulletin] we can obtain an expression for the kinetic plus (electro)magnetic energy budget of a MHD-plasma.

- Express the source/sink term appearing in Poynting's equation as $-\mathbf{v}\cdot(\mathbf{j}\wedge\mathbf{B})/c - \|\mathbf{j}\|^2/\sigma_{\rm e}$.
- Add together the balance equations for the (electro)magnetic and for the kinetic energy of the plasma contained in $\Omega_{\rm t}$.
- Discuss what we can understand as 'dynamo' and 'motor' from the resulting balance equation.

3. The so-called *First Principle of Thermodynamics* for a MHD-plasma.

From the integral expression of the principle of energy conservation in a MHD-plasma (without chemical or nuclear reactions) we can obtain an integral balance equation for the internal –or 'thermal'– energy $U(\Omega_{\mathrm{t}})$.

- Use Poynting's theorem (for a material region) along with the theorem for the kinetic energy to show that
$$\frac{d}{dt}\int_{\Omega_{\mathrm{t}}} \rho\epsilon = \int_{\Omega_{\mathrm{t}}} \hat{\boldsymbol{\sigma}} : \hat{\boldsymbol{D}} - \oiint_{\partial\Omega_{\mathrm{t}}} \boldsymbol{\mathcal{F}} \cdot \mathbf{n} + \int_{\Omega_{\mathrm{t}}} \frac{\|\mathbf{j}\|^2}{\sigma_{\mathrm{e}}}, \tag{2}$$
where $\epsilon(\mathbf{x}, t)$ is the specific internal energy (i.e., per unit mass) and $\boldsymbol{\mathcal{F}}(\mathbf{x}, t)$ is the heat flux density vector.

- Obtain a differential equation from the integral expression (2). This equation is the local or differential form of the *First Principle of Thermodynamics* for a MHD-plasma.

4. Energy equation in the (p, T) representation.

Starting from the *First Principle of Thermodynamics* in differential form, show that the energy equation for a MHD-plasma which is a Newtonian fluid can be written in terms of the thermodynamic variables pressure and temperature in the form
$$\rho c_p \frac{DT}{Dt} - \alpha T \frac{Dp}{Dt} = \Phi_v + \Phi_m - \mathrm{div}\,\boldsymbol{\mathcal{F}}, \tag{3}$$
where Φ_v and Φ_m are, respectively, the viscous and the Ohmic dissipation functions, α is the coefficient of thermal expansion and c_p is the specific heat at constant pressure.

Bulletin of exercises n°6: Force-free magnetic fields.

If the electric currents flow along the magnetic field lines, the equation of magnetostatic equilibrium is

$$\mathbf{j} \wedge \mathbf{B} = 0\,, \tag{1}$$

in which case the Lorentz force vanishes everywhere and hydrostatic equilibrium is independent of the magnetic field.

From Eq. (1) it is immediate that the curl of $\mathbf{B}$ is everywhere parallel to the field $\mathbf{B}$ itself: $\mathbf{rot}\,\mathbf{B} = \alpha\,\mathbf{B}$, where $\alpha(\mathbf{x})$ is a function of position.

Prove the following general properties of force-free fields:

1. Function $\alpha(\mathbf{x})$. Topological properties.

1.1. The function $\alpha(\mathbf{x})$ is constant along every magnetic field line (but differs, in general, from line to line).

1.2. The integral lines of the fields $\mathbf{B}$ and $\mathbf{j}$ lie on surfaces $\alpha(\mathbf{x}) = \text{const.}$

1.3.* If a surface $\alpha(\mathbf{x}) = \text{const.}$ is closed, then it cannot be simply connected.

2. Magnetic energy: The following results are consequences of the restriction imposed on the magnetic energy by the condition that the magnetic field be force-free.

Theorem: If $\mathbf{B}$ is force-free in a region $\mathcal{R}$ with boundary $\partial\mathcal{R}$, the total magnetic energy contained in $\mathcal{R}$ is uniquely determined by the values taken by $\mathbf{B}$ on the boundary $\partial\mathcal{R}$.

Corollary I: It is not possible to have a force-free magnetic field inside a bounded region $\mathcal{R}$ that identically vanishes on the boundary $\partial\mathcal{R}$.

Corollary II: It is not possible to have a force-free magnetic field inside a bounded region $\mathcal{R}$ and such that it is entirely maintained by electric currents confined within $\mathcal{R}$.

3. 'Linear' force-free fields.

In general, $\alpha(\mathbf{x})$ takes different values on different lines of force. In the particular case when $\alpha(\mathbf{x})$ takes the same value everywhere, show that the magnetic field $\mathbf{B}$ satisfies Helmholtz' differential equation, viz.

$$(\nabla^2 + \alpha^2)\,\mathbf{B} = 0\,. \tag{2}$$

The exercise indicated with * is difficult and will not be discussed in the resolution of the exercises.