Appendix A

Self-Adjoint Operators for Adiabatic Oscillations in Stellar Atmospheres [1]

In this appendix we show that adiabatic oscillations in a hydrostatic magnetized atmosphere have zero growth rate. In the adiabatic case, the eigenvalue problem for the square of the frequency, equations (2.10)-(2.13), can be cast in the familiar form:

$$\mathfrak{L}\mathbf{\Psi} = \omega^2 \mathbf{\Psi} \tag{A.1}$$

with the eigenvector

$$\mathbf{\Psi} = \begin{pmatrix} \chi \\ \xi \end{pmatrix} = \begin{pmatrix} iU \\ W \end{pmatrix} \tag{A.2}$$

We have used $\chi = iU$ to absorb all the factors of i. The linear operator $\mathfrak{L}$ then has the form

$$\mathfrak{L}\mathbf{\Psi} = \begin{pmatrix} c^2(k^2\chi + k\frac{d\xi}{dz} + \frac{k}{\gamma H_P}\xi) + a^2(k^2\chi - \frac{d^2\chi}{dz^2}) \\ -c^2(k\frac{d\chi}{dz} + \frac{d^2\xi}{dz^2} + \frac{1}{\gamma}(\frac{1}{{H_P}^2} - \frac{1}{{H_P}^2}\frac{dH_P}{dz} - \frac{1}{H_P H_D})\xi + \frac{\gamma-1}{\gamma H_P}k\chi + \frac{1}{H_P}\frac{d\xi}{dz}) \end{pmatrix} \tag{A.3}$$

[1] *This is the appendix from a Solar Physics paper with A. Kosovichev, E. Spigel, and L. Tao (Birch et al., 2001b). This appendix was mostly my work, though L. Tao helped with some of the writing.*

where $H_D = (\frac{d\log\rho_0}{dz})^{-1}$, $H_P = (\frac{d\log P_0}{dz})^{-1}$ are the density and pressure scale heights (which in general can be functions of depth). Implicit in the definition of this boundary-value eigenvalue problem are the boundary conditions, but we defer their consideration for now. To discuss the spectral properties of the operator $\mathcal{L}$, we need to define an inner product:

$$\langle \mathbf{\Psi_a} = (\chi_a, \xi_a), \mathbf{\Psi_b} = (\chi_b, \xi_b)\rangle = \int_{z_1}^{z_2} dz \rho_0 [\chi_a^* \chi_b + \xi_a^* \xi_b]\,. \tag{A.4}$$

Our treatment holds for arbitrary hydrostatic stratification. The magnetic field does not change the equilibrium because it has no current.

Let us consider the following inner product:

$$\begin{aligned}\langle \mathbf{\Psi_a}, \mathcal{L}\mathbf{\Psi_b}\rangle \;=\; & \int_{z_1}^{z_2} dz \rho_0 c^2 \Big[k^2 \chi_a^* \chi_b + k \chi_a^* \frac{d\xi_b}{dz} + \frac{k}{\gamma H_P}\chi_a^* \xi_b \\ & + \alpha^2 \left(k^2 \chi_a^* \chi_b - \chi_a^* \frac{d^2\chi_b}{dz^2}\right) - k\xi_a^* \frac{d\chi_b}{dz} - \xi_a^* \frac{d^2\xi_b}{dz^2} \\ & - \frac{1}{\gamma}\left(\frac{1}{{H_P}^2} - \frac{1}{{H_P}^2}\frac{dH_P}{dz} + \frac{1}{H_P H_D}\right)\xi_a^* \xi_b \\ & - \frac{\gamma-1}{\gamma H_P} k \xi_a^* \chi_b - \frac{1}{H_P}\xi_a^* \frac{d\xi_b}{dz}\Big]\end{aligned}$$

where $\alpha^2 = \frac{a^2}{c^2}$.

Partial integration on the $\alpha^2 \chi_a^* \frac{d^2\chi_b}{dz^2}$, $\xi_a^* \frac{d^2\xi_b}{dz^2}$ and $k\xi_a^* \frac{d\chi_b}{dz}$ terms gives:

$$\begin{aligned}\langle \mathbf{\Psi_a}, \mathcal{L}\mathbf{\Psi_b}\rangle \;=\; & \int_{z_1}^{z_2} dz \rho_0 c^2 \Big[2k\left(\chi_a^* \frac{d\xi_b}{dz} + \frac{d\xi_a^*}{dz}\chi_b\right) + \frac{2k}{\gamma H_P}\left(\chi_a^* \xi_b + \xi_a^* \chi_b\right) \\ & + \alpha^2 \frac{d\chi_a^*}{dz}\frac{d\chi_b}{dz} + \frac{d\xi_a^*}{dz}\frac{d\xi_b}{dz} + (1+\alpha^2)k^2 \chi_a^* \chi_b \\ & - \frac{1}{\gamma}\left(\frac{1}{H_P H_T} - \frac{1}{H_P^2}\frac{dH_P}{dz}\right)\xi_a^* \xi_b\Big] + \mathcal{B}\end{aligned}$$

where $\mathcal{B}$ represents the boundary contribution:

$$\mathcal{B} = -\left[\rho_0 c^2 \left(\xi_a^* \frac{d\xi_b}{dz} + \alpha^2 \chi_a^* \frac{d\chi_b}{dz} + \xi_a^* \chi_b\right)\right]_{z_1}^{z_2}. \tag{A.5}$$

Each of the three terms in the $\mathcal{B}$ have simple physical interpretations. The

α^2 term depends only on the horizontal motions and corresponds to the magnetic energy flux through the boundaries. By choosing either χ or $d\chi/dz$ to vanish at the boundaries, the magnetic term can be removed. The other two terms combine as the total mechanical energy flux through the boundaries. They can be removed by setting $\xi = 0$ on the boundaries. For our numerical study here, we have set $\xi = 0$ and $d\chi/dz = 0$ at both the top and bottom boundaries, and $\mathcal{B}$ is identically zero. These mathematically convenient boundary conditions therefore correspond to physically consistent boundary conditions.

Whenever the boundary contributions can be neglected, the operator $\mathcal{L}$ with the given inner product and boundary conditions is self-adjoint:

$$\langle \mathbf{\Psi_a}, \mathcal{L}\mathbf{\Psi_b} \rangle = \langle \mathcal{L}\mathbf{\Psi_a}, \mathbf{\Psi_b} \rangle \tag{A.6}$$

and its eigenvalues ω^2 are real.

When the boundary terms are non-zero, the linear operator is no longer self-adjoint, and the eigenvalues ω^2 are in general complex. Physically we see that this instability is forced via the transfer of energy through the boundaries since the operator is otherwise self-adjoint in the bulk of the atmosphere.

Appendix B

Definition of Travel Time[1]

This appendix is a detailed derivation of equation (3.50), which gives the linearized relationship between perturbations to the travel time and perturbations to the cross-correlation. According to equation (3.45) the travel times $\tau_+(\mathbf{1},\mathbf{2})$ and $\tau_-(\mathbf{1},\mathbf{2})$ are the time lags which minimize the functions

$$X_\pm(\mathbf{1},\mathbf{2},t) = \int_{-\infty}^{\infty} \mathrm{d}t' \; f(\pm t') \left[C(\mathbf{1},\mathbf{2},t') - C^{\mathrm{ref}}(\mathbf{1},\mathbf{2},t' \mp t)\right]^2 . \qquad \text{(B.1)}$$

As a result the time derivatives of $X_\pm$ evaluated at $\tau_\pm$ are zero:

$$\dot{X}_\pm(\mathbf{1},\mathbf{2},\tau_\pm) = 0 . \qquad \text{(B.2)}$$

Notice that $\dot{X}$ does not involve a time derivative of the observed cross-correlation C. In order to obtain the travel-time perturbations $\delta\tau_\pm$ we need to linearize around the zero-order travel times $\tau^0_\pm$, which are defined by

$$\tau^0_\pm(\mathbf{1},\mathbf{2}) = \underset{t}{\mathrm{argmin}}\{\; X^0_\pm(\mathbf{1},\mathbf{2},t)\;\} . \qquad \text{(B.3)}$$

The functions $X^0_\pm$ refer to equation (B.1) evaluated for $C = C^0$, where C^0 is the zero-order cross-correlation in the reference model. Linearizing equation (B.2) about

[1] *This appendix is from (Gizon & Birch, 2002)*

$\tau_\pm = \tau^0_\pm$ gives

$$\delta\tau_\pm(\mathbf{1},\mathbf{2}) = -\frac{\delta\dot{X}_\pm(\mathbf{1},\mathbf{2},\tau^0_\pm)}{\ddot{X}^0_\pm(\mathbf{1},\mathbf{2},\tau^0_\pm)}\,. \tag{B.4}$$

The functions $\delta\dot{X}_\pm$ are given by

$$\delta\dot{X}_\pm(\mathbf{1},\mathbf{2},t) = \pm 2\int_{-\infty}^{\infty} \mathrm{d}t'\; f(\pm t')\dot{C}^{\mathrm{ref}}(\mathbf{1},\mathbf{2},t'\mp t)\delta C(\mathbf{1},\mathbf{2},t')\,. \tag{B.5}$$

We can then compute $\ddot{X}^0(\tau^0_\pm)$ by straightforward differentiation of equation (B.1). The result for $\delta\tau_\pm(\mathbf{1},\mathbf{2})$ is thus

$$\delta\tau_\pm(\mathbf{1},\mathbf{2}) = \int_{-\infty}^{\infty} \mathrm{d}t\; W_\pm(\mathbf{1},\mathbf{2},t)\delta C(\mathbf{1},\mathbf{2},t)\,, \tag{B.6}$$

with

$$W_\pm(t) = \pm\frac{1}{D} f(\pm t)\dot{C}^{\mathrm{ref}}(t\mp\tau^0_\pm) \tag{B.7}$$

where

$$D = \int_{-\infty}^{\infty} \mathrm{d}t'\; \left[f(\pm t')C^0(t')\ddot{C}^{\mathrm{ref}}(t'\mp\tau^0_\pm) \pm \dot{f}(\pm t')C^{\mathrm{ref}}(t'\mp\tau^0_\pm)\dot{C}^{\mathrm{ref}}(t'\mp\tau^0_\pm)\right]. \tag{B.8}$$

We have suppressed the spatial arguments **1** and **2** in the above equation for the sake of readability. This is the general linearized result for arbitrary C^{ref} and f. The only assumption is that the perturbation to the cross-correlation is small compared to the zero-order cross-correlation. Note that we have not written an explicit expression for $\tau^0_\pm$, which needs to be computed numerically by minimizing $X^0_\pm(t)$ (eq. [B.3]).

In the case where C^{ref} and C^0 are even in time, $\tau^0_+ = \tau^0_-$. For the choice $C^{\mathrm{ref}} = C^0$, the zero-order travel times are both zero, $\tau^0_\pm = 0$. This choice is recommended if a theoretical model is available to the observer. With $C^{\mathrm{ref}} = C^0$ the weight functions $W_\pm$ simplify to:

$$W_\pm(\mathbf{1},\mathbf{2},t) = \frac{\mp f(\pm t)\dot{C}^0(\mathbf{1},\mathbf{2},t)}{\int_{-\infty}^{\infty}\mathrm{d}t'\; f(\pm t')[\dot{C}^0(\mathbf{1},\mathbf{2},t')]^2}\,. \tag{B.9}$$

In the example presented in Section 3.4.2 we choose $C^{\mathrm{ref}} = C^0$ and $f(t) = \mathrm{Hea}(t)$.

Appendix C

Fourier Conventions[1]

This appendix outlines the Fourier conventions that are used throughout this dissertation. Given a function $q(\boldsymbol{x}, t)$, of horizontal position $\boldsymbol{x}$ and time t, we employ the convention that the function $q(\boldsymbol{x}, t)$ and its Fourier transform $\check{q}(\boldsymbol{k}, \omega)$ are related by

$$q(\boldsymbol{x}, t) = \iint_{-\infty}^{\infty} \mathrm{d}\boldsymbol{k} \int_{-\infty}^{\infty} \mathrm{d}\omega \, e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \mathrm{i}\omega t} \, \check{q}(\boldsymbol{k}, \omega) \,, \tag{C.1}$$

$$\check{q}(\boldsymbol{k}, \omega) = \frac{1}{(2\pi)^3} \iint_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x} \int_{-\infty}^{\infty} \mathrm{d}t \, e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} + \mathrm{i}\omega t} \, q(\boldsymbol{x}, t) \,, \tag{C.2}$$

where $\boldsymbol{k}$ is a two-dimensional horizontal wave vector and ω is the angular frequency. We commonly use the same symbol for q and $\check{q}$: the arguments make clear whether the function or its transform is intended. We use the notation $q(k, \omega)$ when $q(\boldsymbol{k}, \omega)$ only depends on the magnitude of $\boldsymbol{k}$, not its direction, for example in the filter function $F(k, \omega)$. We note that for functions $q(\boldsymbol{x}, t)$ which do not vanish at large $\|\boldsymbol{x}\|$ or $|t|$ the Fourier transform is not defined. In particular there is a problem for the case when the observable is not windowed in space or time. In such a case, $q(\boldsymbol{k}, \omega)$ is intended to mean the Fourier transform of the function $q(\boldsymbol{x}, t)$ truncated to zero for $|t| > T/2$ and $\|\boldsymbol{x}\| > \sqrt{A/\pi}$, where the time interval T and the area A are both large and finite. This modification enables us to refer to the Fourier transform of a stationary/homogeneous random function (cf. Yaglom, 1962, for a rigorous formalism).

[1] *This appendix is from (Gizon & Birch, 2002)*

When a function of four arguments, $Q(\boldsymbol{x}, t; \boldsymbol{x}', t')$, depends only on the separations $\boldsymbol{x} - \boldsymbol{x}'$ and $t - t'$ (translation invariance), we use the following conventions:

$$\begin{aligned} Q(\boldsymbol{x} - \boldsymbol{x}', t - t') &= Q(\boldsymbol{x}, t; \boldsymbol{x}', t')\,, && \text{(C.3)} \\ Q(\boldsymbol{k}, \omega) &= \frac{1}{(2\pi)^3} \iint_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x} \int_{-\infty}^{\infty} \mathrm{d}t\, e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} + \mathrm{i}\omega t}\, Q(\boldsymbol{x}, t)\,. && \text{(C.4)} \end{aligned}$$

The above conventions are employed, in our example, for the functions $m^0(k, \omega)$, $\boldsymbol{G}(\boldsymbol{k}, \omega; z)$, and $\mathcal{G}(k, \omega)$.

Finally, we recall the relations

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}t\, e^{\mathrm{i}\omega t} &= 2\pi\, \delta_{\mathrm{D}}(\omega)\,, && \text{(C.5)} \\ \iint_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x}\, e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} &= (2\pi)^2\, \delta_{\mathrm{D}}(\boldsymbol{k})\,, && \text{(C.6)} \end{aligned}$$

which are very useful in rewriting the kernels in Fourier space (Appendix D).

Appendix D

Two-Dimensional Travel-Time Sensitivity Kernel [1]

In this appendix we derive surface gravity wave travel-time kernels, $K^a_\pm$ and $K^\gamma_\pm$, for perturbations to local source strength and damping rate respectively. These kernels connect travel-times perturbations, $\delta\tau_\pm$, to perturbations to the model:

$$\delta\tau_\pm(\mathbf{1},\mathbf{2}) = \int_{(A)} \mathrm{d}\boldsymbol{r}\, \frac{\delta a(\boldsymbol{r})}{a}\, K^a_\pm(\mathbf{1},\mathbf{2};\boldsymbol{r}) + \int_{(A)} \mathrm{d}\boldsymbol{r}\, \frac{\delta\gamma(\boldsymbol{r})}{\gamma}\, K^\gamma_\pm(\mathbf{1},\mathbf{2};\boldsymbol{r})\,. \tag{D.1}$$

Here $\delta a(\boldsymbol{r})/a$ is the local fractional change in the source strength and $\delta\gamma(\boldsymbol{r})/\gamma$ the local fractional change in damping rate. The spatial integral $\int_{(A)} \mathrm{d}\boldsymbol{r}$ is a two-dimensional integral taken over all points $\boldsymbol{r}$ on the surface $z = 0$. From section 3.4.1 we know that in order to compute kernels we first need to write the perturbation to the cross-correlation in terms of the functions $\mathcal{C}^a$ and $\mathcal{C}^\gamma$ (see eq. [3.69]):

$$\delta C(\mathbf{1},\mathbf{2},t) = \int_{(A)} \mathrm{d}\boldsymbol{r}\, \frac{\delta a(\boldsymbol{r})}{a}\, \mathcal{C}^a(\mathbf{1},\mathbf{2},t;\boldsymbol{r}) + \int_{(A)} \mathrm{d}\boldsymbol{r}\, \frac{\delta\gamma(\boldsymbol{r})}{\gamma}\, \mathcal{C}^\gamma(\mathbf{1},\mathbf{2},t;\boldsymbol{r})\,. \tag{D.2}$$

The general expression for $\delta C(\mathbf{1},\mathbf{2},t)$ is given by equations (3.65), (3.66), and (3.67). In our example, however, the superscripts on the Green's function can be dropped as the source S is scalar. To obtain $\mathcal{C}^a$, we use equation (3.67) for $\mathcal{C}_S$ and the definition of the source perturbation δM (eqs. [3.89] and [3.93]). After integrations by parts

[1] *This is appendix is from (Gizon & Birch, 2002).*

on the source variables in the right-hand side of equation (3.67) and the change of variables $\boldsymbol{r} = (\boldsymbol{s} + \boldsymbol{s}')/2$ and $\boldsymbol{u} = \boldsymbol{s} - \boldsymbol{s}'$, we obtain

$$\begin{aligned} \mathcal{C}^a(\mathbf{1}, \mathbf{2}, t; \boldsymbol{r}) &= \frac{1}{T} \int \mathrm{d}t' \, \mathrm{d}t_\mathrm{s} \, \mathrm{d}t'_\mathrm{s} \, \mathrm{d}\boldsymbol{u} \; m^0(\boldsymbol{u}, t_\mathrm{s} - t'_\mathrm{s}) \\ &\quad \times \mathcal{G}^{\boldsymbol{\Pi}}(\mathbf{1} - \boldsymbol{r} - \boldsymbol{u}/2, t' - t_\mathrm{s}) \, \mathcal{G}^{\boldsymbol{\Pi}}(\mathbf{2} - \boldsymbol{r} + \boldsymbol{u}/2, t' - t'_\mathrm{s} + t) \, . \end{aligned} \tag{D.3}$$

The function $\mathcal{C}^\gamma$ is obtained from equation (3.66) with $\delta\mathcal{L}$ defined by equations (3.86) and (3.98). After integrations by parts on the source variables, and a partial integration on the variable $\boldsymbol{r}$, the result is

$$\begin{aligned} \mathcal{C}^\gamma(\mathbf{1}, \mathbf{2}, t; \boldsymbol{r}) &= \frac{1}{2\pi T} \int \mathrm{d}t' \, \mathrm{d}t'' \, \mathrm{d}\boldsymbol{s} \, \mathrm{d}t_\mathrm{s} \, \mathrm{d}\boldsymbol{s}' \, \mathrm{d}t'_\mathrm{s} \, \mathrm{d}\bar{t} \; \Gamma^0(t'' - \bar{t}) \\ &\quad \times m^0(\boldsymbol{s} - \boldsymbol{s}', t_\mathrm{s} - t'_\mathrm{s}) \, \boldsymbol{\nabla}^2_\mathrm{h} \dot{\boldsymbol{G}}_\mathrm{h}(\boldsymbol{r} - \boldsymbol{s}, \bar{t} - t_\mathrm{s}) \\ &\quad \cdot [\boldsymbol{\alpha} + \boldsymbol{\beta}] \; , \end{aligned} \tag{D.4}$$

with

$$\boldsymbol{\alpha} = \mathcal{G}^{\boldsymbol{\Pi}}(\mathbf{1} - \boldsymbol{s}', t' - t'_\mathrm{s}) \, \boldsymbol{\nabla}_\mathrm{h} \dot{\mathcal{G}}(\mathbf{2} - \boldsymbol{r}, t' + t - t'') \tag{D.5}$$

$$\boldsymbol{\beta} = \mathcal{G}^{\boldsymbol{\Pi}}(\mathbf{2} - \boldsymbol{s}', t' + t - t'_\mathrm{s}) \, \boldsymbol{\nabla}_\mathrm{h} \dot{\mathcal{G}}(\mathbf{1} - \boldsymbol{r}, t' - t'') \tag{D.6}$$

where $\boldsymbol{G}_\mathrm{h}$ denotes the two horizontal components of the vector $\boldsymbol{G}$. In the space-time domain these integrals are quite complicated to compute. They, however, are greatly simplified when written in terms of the Fourier transforms of the various functions:

$$\begin{aligned} \mathcal{C}^a(\mathbf{1}, \mathbf{2}, t; \boldsymbol{r}) &= (2\pi)^4 \int \mathrm{d}\omega \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\boldsymbol{k}' \; e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Delta}_1 - \mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{\Delta}_2 - \mathrm{i}\omega t} \, m^{0*}[(\boldsymbol{k} + \boldsymbol{k}')/2, \omega] \\ &\quad \times \mathcal{G}^{\boldsymbol{\Pi}*}(\boldsymbol{k}, \omega) \, \mathcal{G}^{\boldsymbol{\Pi}}(\boldsymbol{k}', \omega) \, , \end{aligned} \tag{D.7}$$

$$\begin{aligned} \mathcal{C}^\gamma(\mathbf{1}, \mathbf{2}, t; \boldsymbol{r}) &= (2\pi)^7 \int \mathrm{d}\omega \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\boldsymbol{k}' \; \left(e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Delta}_1 - \mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{\Delta}_2 - \mathrm{i}\omega t} + e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Delta}_2 - \mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{\Delta}_1 + \mathrm{i}\omega t} \right) \\ &\quad \times \Gamma^0(\omega) m^0(\boldsymbol{k}, \omega) G^\Pi(\boldsymbol{k}, \omega) \mathcal{G}^{\boldsymbol{\Pi}*}(\boldsymbol{k}, \omega) \; \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}' \; \mathcal{G}^{\boldsymbol{\Pi}}(\boldsymbol{k}', \omega)/k' \, . \end{aligned} \tag{D.8}$$

We have used the definitions

$$\mathcal{G}^{\Pi}(\boldsymbol{k},\omega) = F(\boldsymbol{k},\omega) G^{\Pi}(\boldsymbol{k},\omega)\,, \tag{D.9}$$

$$G^{\Pi}(\boldsymbol{k},\omega) = \mathrm{i}\omega k^2 G_z(\boldsymbol{k},\omega; z=0)\,, \tag{D.10}$$

and the identity $\boldsymbol{G}_{\mathrm{h}}(\boldsymbol{k},\omega) = \mathrm{i}\hat{\boldsymbol{k}} G_z(\boldsymbol{k},\omega)$ resulting from equation (3.110). The Green's function $G_z(k,\omega)$ is the $\hat{\boldsymbol{z}}$ component of $\boldsymbol{G}$, given by equation (3.110).

With the assumption that m^0 is independent of $\boldsymbol{k}$, the above expressions can be simplified to

$$\mathcal{C}^{a}(\mathbf{1},\mathbf{2},\omega;\boldsymbol{r}) = m^0(\omega)\, \mathrm{I}^*(\Delta_1,\omega)\, \mathrm{I}(\Delta_2,\omega)\,, \tag{D.11}$$

$$\mathcal{C}^{\gamma}(\mathbf{1},\mathbf{2},\omega;\boldsymbol{r}) = m^0(\omega)\, \hat{\boldsymbol{\Delta}}_1 \cdot \hat{\boldsymbol{\Delta}}_2 \times[\mathrm{I\!I}(\Delta_1,\omega)\, \mathrm{I\!I\!I}(\Delta_2,\omega) + \mathrm{I\!I}(\Delta_2,\omega)\, \mathrm{I\!I\!I}^*(\Delta_1,\omega)]\,. \tag{D.12}$$

The integrals I, II, and III are given by

$$\mathrm{I}(d,\omega) = (2\pi)^3 \int_0^\infty k\mathrm{d}k\; J_0(kd) \mathcal{G}^{\Pi}(k,\omega)\,, \tag{D.13}$$

$$\mathrm{I\!I}(d,\omega) = (2\pi)^6\, \Gamma^0(\omega) \int_0^\infty k\mathrm{d}k\; J_1(kd) G^{\Pi}(k,\omega) \mathcal{G}^{\Pi *}(k,\omega)\,, \tag{D.14}$$

$$\mathrm{I\!I\!I}(d,\omega) = (2\pi)^3 \int_0^\infty \mathrm{d}k\; J_1(kd) \mathcal{G}^{\Pi}(k,\omega)\,. \tag{D.15}$$

The kernels for source strength and damping are then obtained from

$$K^{a,\gamma}_{\pm}(\mathbf{1},\mathbf{2},\boldsymbol{r}) = 4\pi \mathrm{Re} \int_0^\infty \mathrm{d}\omega\; W^*_{\pm}(\mathbf{1},\mathbf{2},\omega) \mathcal{C}^{a,\gamma}(\mathbf{1},\mathbf{2},\omega;\boldsymbol{r}) \tag{D.16}$$

with $W^*_{\pm}(\mathbf{1},\mathbf{2},\omega)$ given equation (3.106). The kernels, in terms of the integrals I, II, and III, are reported in the main body of the text (eqs. [3.118] and [3.120]).

Appendix E

Two-Dimensional Single-Source Kernels for the Damping Rate[1]

This appendix is a derivation of $K_+^{\gamma,\mathrm{ss}}$, which first appears in the main text in equation (3.126). In the single-source picture, we want to derive a kernel $K_+^{\gamma,\mathrm{ss}}$ which provides an integral relationship between the one-way travel time $\delta\tau_+^{\mathrm{ss}}$ (eq. [3.124]) and the local damping perturbation $\delta\gamma(\boldsymbol{r})/\gamma$, i.e.

$$\delta\tau_+^{\mathrm{ss}}(\mathbf{1},\mathbf{2}) = \int_{(A)} \mathrm{d}\boldsymbol{r}\, \frac{\delta\gamma(\boldsymbol{r})}{\gamma}\, K_+^{\gamma,\mathrm{ss}}(\mathbf{1},\mathbf{2};\boldsymbol{r})\,. \tag{E.1}$$

We first rewrite the single-source definition of travel time (eq. [3.124]) in terms of the temporal Fourier transform of the signal observed at point $\mathbf{2}$:

$$\delta\tau_+^{\mathrm{ss}}(\mathbf{1},\mathbf{2}) = -\frac{\mathrm{Re}\int_0^\infty \mathrm{d}\omega\; \mathrm{i}\omega\, \phi^{0*}(\mathbf{2},\omega)\, \delta\phi(\mathbf{2},\omega)}{\int_0^\infty \mathrm{d}\omega\; \omega^2\, |\phi^0(\mathbf{2},\omega)|^2}\,. \tag{E.2}$$

[1] *This appendix is from (Gizon & Birch, 2002)*

Given the pressure source $\rho\Theta$, located at point $\mathbf{1}$, and defined by equation (3.125), the zero- and first-order signals observed at $\mathbf{2}$ are

$$\phi^0(\mathbf{2},\omega) = (2\pi)^4 \int_0^\infty k\mathrm{d}k\; J_0(k\Delta)\mathcal{G}^{\mathbf{\Pi}}(k,\omega)\Theta(k,\omega)\,, \tag{E.3}$$

$$\delta\phi(\mathbf{2},\omega) = (2\pi)^4\,\Gamma^0(\omega)\int_{(A)} \mathrm{d}\boldsymbol{r}\;\frac{\delta\gamma(\boldsymbol{r})}{\gamma}\int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}'\;e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\Delta}_1-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{\Delta}_2} \tag{E.4}$$
$$\times G^{\mathbf{\Pi}}(\boldsymbol{k},\omega)\Theta(\boldsymbol{k},\omega)\;\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'\;\mathcal{G}^{\mathbf{\Pi}}(\boldsymbol{k}',\omega)/k'\,.$$

Using equation (E.2) we obtain the damping kernel $K_+^{\gamma,\mathrm{ss}}$ in the form

$$K_+^{\gamma,\mathrm{ss}}(\mathbf{1},\mathbf{2};\boldsymbol{r}) = \frac{\int_0^\infty \mathrm{d}\omega\;\omega^2|\phi^0(\mathbf{2},\omega)|^2\;\mathcal{K}_+^{\gamma,\mathrm{ss}}(\mathbf{1},\mathbf{2};\boldsymbol{r};\omega)}{\int_0^\infty \mathrm{d}\omega\;\omega^2|\phi^0(\mathbf{2},\omega)|^2}\,, \tag{E.5}$$

with the function $\mathcal{K}_+^{\gamma,\mathrm{ss}}$ (single-frequency kernel) defined by

$$\mathcal{K}_+^{\gamma,\mathrm{ss}}(\mathbf{1},\mathbf{2};\boldsymbol{r};\omega) = \hat{\boldsymbol{\Delta}}_1\cdot\hat{\boldsymbol{\Delta}}_2\;\;\mathrm{Im}\left[\frac{\mathbb{IV}(\Delta_1,\omega)\;\mathbb{III}(\Delta_2,\omega)}{\omega\;\phi^0(\mathbf{2},\omega)}\right]\,. \tag{E.6}$$

In the above equation, the function $\mathbb{IV}$ is a one-dimensional integral given by

$$\mathbb{IV}(d,\omega) = (2\pi)^3\,\Gamma^0(\omega)\int_0^\infty k\mathrm{d}k\;J_1(kd)G^{\mathbf{\Pi}}(k,\omega)\Theta(k,\omega)\,, \tag{E.7}$$

and the function $\mathbb{III}$ denotes the integral already defined by equation (D.15). Notice from equation (E.5) that the kernel $K_+^{\gamma,\mathrm{ss}}$ is a frequency average of $\mathcal{K}_+^{\gamma,\mathrm{ss}}$ weighted by $\omega^2|\phi^0(\mathbf{2},\omega)|^2$.

In order to compute the kernel we have to make a choice for the source spectrum, $\Theta(k,\omega)$. In general, this is difficult without a priori knowledge of the zero-order cross-correlation. When comparing the definition of travel time of Appendix B with the single-source definition (eq. [3.124]), we find that a good match between the two definitions is obtained when $\phi^0(\mathbf{2},t)$ looks like $\mathrm{Hea}(t)C^0(\mathbf{1},\mathbf{2},t)$. This condition is best met when

$$\Theta(k,\omega) = -\frac{k\,F(k,\omega)\,m^0(k,\omega)}{2\Gamma^0(\omega)}\,. \tag{E.8}$$

Note that the filter function $F(k,\omega)$ appears in equation (E.8). The kernel $K_+^{\gamma,\mathrm{ss}}$, shown in Figure 3.15, was computed using this choice.

Appendix F

Derivation of Deep-Focusing Kernels

This appendix gives a detailed derivation of travel-time kernels for the deep-focusing time-distance method discussed in section 4.3. The calculations here are all straightforward applications of the basic theory developed in section 3.4.1.

Section 4.3.2 gives the definition of the deep-focusing cross-correlation and a discussion of the measurement of travel times from these cross-correlations. The purpose of this appendix is compute the travel time kernels $K(r)$, which connect a sound speed perturbation that is only a function of radius with a perturbation to a deep-focusing travel time. The kernels should satisfy

$$\delta\tau = \int_0^{R_\odot} \mathrm{d}r \; K(r) \frac{\delta c(r)}{c(r)} \, . \tag{F.1}$$

In order to compute these kernels we follow the basic recipe presented in section 3.4.1. In the remainder of this introductory section we define the deep-focusing cross-correlation and discuss the measurement of travel times. The Green's function is discussed in section F.1. Section F.2 gives a derivation of the zero-order cross-correlation. The power spectrum is discussed in section F.3. The final section, F.4, gives the derivation of the kernels.

The deep-focusing cross-correlation is the time-shifted and windowed average of the individual cross-correlations

$$\bar{C}(t) = \sum_{i=1}^{N} f(t - t_i) C_i(t - t_i) \,. \tag{F.2}$$

In the above equation $\bar{C}(t)$ is the deep-focusing cross-correlation and C_i are the single-distance correlations. The index i gives the distance at which the cross-correlation C_i is computed. The times t_i are the nominal travel times; they depend on the distance index i. The function $f(t)$ is the window function, taken to be one for $|t| < 10$ min and zero elsewhere. The number of distances that are used in the deep-focusing average is given by N.

The zero-order deep-focusing cross-correlation is

$$\bar{C}^0(t) = \sum_{i=1}^{N} f(t - t_i) C_i^0(t - t_i) \tag{F.3}$$

and the first-order perturbation to the deep-focusing cross-correlation is

$$\delta\bar{C}(t) = \sum_{i=1}^{N} f(t - t_i) \delta C_i(t - t_i) \,. \tag{F.4}$$

Throughout this appendix the superscript 0 will be used to denote background quantities, and a preceding δ to denote first-order perturbations.

As in section 3.4.1, we measure travel times by minimizing a function X,

$$\tau = \underset{t}{\operatorname{argmin}} \; X(t) \,, \tag{F.5}$$

where

$$X(t) = \int \mathrm{d}t' \; \left[\bar{C}(t') - C^{\mathrm{ref}}(t' - t)\right]^2 \,. \tag{F.6}$$

Notice that a reference cross-correlation, C^{ref}, has been introduced. In the case of the deep-focusing study in this dissertation the average deep-focusing cross-correlation over longitude was employed as the reference cross-correlation.

According to the recipe, we now need to obtain a linear relationship between perturbations to the cross-correlations and the travel time. We are looking for the argument of X where the derivative of X is zero. The time derivative of X is

$$\dot{X}(\tau) \propto \int \mathrm{d}t \ \left[\bar{C}(t) - C^{\mathrm{ref}}(t-\tau)\right] \dot{C}^{\mathrm{ref}}(t-\tau)\,. \tag{F.7}$$

In the unperturbed case $\bar{C} = \bar{C}^0$ and X is maximum when $\tau = \tau^0$, which we take as the definition for the time lag τ_0. The zero order time lag τ^0 thus satisfies

$$0 = \int \mathrm{d}t \ \left[\bar{C}^0(t) - C^{\mathrm{ref}}(t-\tau^0)\right] \dot{C}^{\mathrm{ref}}(t-\tau^0)\,. \tag{F.8}$$

If we perturb $\dot{X}(\tau)$ around $\tau = \tau^0$ in equation (F.7) and set the result equal to zero the result is

$$\begin{aligned} 0 \ = \ & \int \mathrm{d}t \ \left[\bar{C}^0(t) + \delta\bar{C}(t) - C^{\mathrm{ref}}(t-\tau^0) + \delta\tau\dot{C}^{\mathrm{ref}}(t-\tau^0)\right] \\ & \times\left[\dot{C}^{\mathrm{ref}}(t-\tau^0) - \delta\tau\ddot{C}^{\mathrm{ref}}(t-\tau^0)\right] \end{aligned} \tag{F.9}$$

The above equation can simplified by using equation (F.8). The result is

$$\delta\tau = \frac{\int \mathrm{d}t \ \delta\bar{C}(t)\dot{C}^{\mathrm{ref}}(t-\tau^0)}{\int \mathrm{d}t \ \bar{C}^0(t)\ddot{C}^{\mathrm{ref}}(t-\tau^0)}\,. \tag{F.10}$$

We have used the assumption that C^{ref} vanishes at the ends of the time window, which is reasonable. This is the result that we had set out to obtain, a linear relationship between the perturbation to the cross-correlation and the perturbation to the deep-focusing travel time.

F.1 Green's Function

The spatially filtered zero order Green's function is (Dahlen & Tromp, 1998)

$$\begin{aligned} \mathcal{G}(\boldsymbol{x}, \boldsymbol{x}_s, t) \ = \ & \sum_{nlm} F_l U_{nl}(R_\odot) D_{nl}(\boldsymbol{x}_s) \\ & \times Y_{lm}(\boldsymbol{x}) Y_{lm}(\boldsymbol{x}_s) \cos(\omega_{nl} t) \exp(-\gamma_{nl} t) \mathrm{Hea}(t)\,. \end{aligned} \tag{F.11}$$

This Green's function gives the radial velocity response (which we are assuming is the signal) at location $\boldsymbol{x}$ and time t to a monopole impulsive source at the solar surface at $\boldsymbol{x}_s$ at time zero. The γ_{nl} are the mode damping rates. For this work we assume that the filter, $\mathcal{F}$, acts on spherical harmonics as

$$\mathcal{F}\left\{Y_{lm}(\theta,\phi)\right\} = F_l Y_{lm}(\theta,\phi)\,. \tag{F.12}$$

Equation (F.11) for the Green's function is only exact when the eigenfunctions form a complete basis. As we have seen throughout this dissertation, once we have the zero order Green's function we can obtain the zero-order cross-correlation by making some assumption about the statistics of the wave sources.

F.2 Zero-Order Cross-Correlation

From section 3.4.1 we know that the zero-order cross-correlation can be obtained from the Green's function

$$\begin{aligned} C^0(\mathbf{1},\mathbf{2},t) &= \frac{1}{T}\int \mathrm{d}t'\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}t_\mathrm{s}\,\mathrm{d}\boldsymbol{s}'\,\mathrm{d}t'_\mathrm{s}\; M^0(\boldsymbol{s},t_\mathrm{s};\boldsymbol{s}',t'_\mathrm{s}) \\ &\quad \times \mathcal{G}(\mathbf{1},\boldsymbol{s},t'-t_\mathrm{s})\,\mathcal{G}(\mathbf{2},\boldsymbol{s}',t'+t-t'_\mathrm{s})\,. \end{aligned} \tag{F.13}$$

For the sake of computational simplicity we assume that the sources are uncorrelated in space and time, so that the source covariance can be written as

$$M^0 = A(s)\delta_D(\boldsymbol{s}-\boldsymbol{s}')\delta_D(t_\mathrm{s}-t'_\mathrm{s})\,. \tag{F.14}$$

Here $s = |\boldsymbol{s}|$ is the source depth variable and the function $A(r)$ gives the distribution of the square of the source strength with depth.

With equation (F.14) for the source covariance, the cross-correlation becomes

$$\begin{aligned} C^0(\mathbf{1},\mathbf{2},t) &= \frac{1}{T}\int \mathrm{d}t'\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}t_\mathrm{s}\; A(|\boldsymbol{s}|) \\ &\quad \times \mathcal{G}(\mathbf{1},\boldsymbol{s},t'-t_\mathrm{s})\,\mathcal{G}(\mathbf{2},\boldsymbol{s},t'+t-t_\mathrm{s})\,. \end{aligned} \tag{F.15}$$

By plugging equation (F.11) into the above equation and doing some algebra,

$$\begin{aligned} C^0(\mathbf{1},\mathbf{2},t) &= \frac{1}{T}\sum_{nlm}\int \mathrm{d}t'\, s^2\mathrm{d}s\, \mathrm{d}t_\mathrm{s}\; A(s) \\ &\times F_l^2 U_{nl}^2(R_\odot) Y_{lm}(\mathbf{1}) Y_{lm}(\mathbf{2}) D_{nl}^2(s) \\ &\times \cos\left(\omega_{nl}(t'-t_\mathrm{s})\right) \exp\left(-\gamma_{nl}(t'-t_\mathrm{s})\right)\mathrm{Hea}(t-t_\mathrm{s}) \\ &\times \cos\left(\omega_{nl}(t'+t-t_\mathrm{s})\right) \exp\left(-\gamma_{nl}(t'+t-t_\mathrm{s})\right)\mathrm{Hea}(t'+t-t_\mathrm{s})\,. \end{aligned} \tag{F.16}$$

The time integrals over t' and t_s can be done in the small damping limit, and the sum over m can be done analytically to obtain

$$\begin{aligned} C^0(\mathbf{1},\mathbf{2},t) &= \sum_{nl}\frac{2l+1}{4\pi}\int s^2\mathrm{d}s\; A(s) F_l^2 U_{nl}^2(R_\odot) D_{nl}^2(s) P_l(\cos\Delta) \\ &\times \frac{\exp\left(-\gamma_{nl}|t|\right)}{4\gamma_{nl}}\cos\left(\omega_{nl}t\right)\,. \end{aligned} \tag{F.17}$$

In the above equation Δ is the distance between $\mathbf{1}$ and $\mathbf{2}$ and P_l are the Legendre polynomials. If we make the further simplification that the wave sources are all located at a particular depth s then the cross-correlation simplifies to

$$\begin{aligned} C^0(\mathbf{1},\mathbf{2},t) &= \sum_{nl}\frac{2l+1}{4\pi} F_l^2 U_{nl}^2(R_\odot) D_{nl}^2(s) P_l(\cos\Delta) \\ &\times \frac{\exp\left(-\gamma_{nl}|t|\right)}{4\gamma_{nl}}\cos\left(\omega_{nl}t\right)\,. \end{aligned} \tag{F.18}$$

Notice that we have nelgected to include the factor $s^2A(s)$. This is because an overall amplitude to the cross-correlation that doesn't depend on distance or time is not relevant to the computation of travel-time kernels. Equation (F.19) gives the zero order cross-correlation in the background solar model.

F.3 Power Spectrum

The calculation of the power spectrum is, in a sense, a digression from the main task of this appendix. Knowledge of the power spectrum for a particular source model is very useful however, as it provides a constraint on the source covariance matrix

M. The calculation here proceeds in much the same manner as the derivation of equation (3.113).

We define the power spectrum to be (Dahlen & Tromp, 1998)

$$P_l(\omega) = \frac{1}{2l+1} \sum_{m=-l}^{l} E\left[\phi_{lm}^*(\omega)\phi_{lm}(\omega)\right] \tag{F.19}$$

where ϕ is the observable, in this case radial velocity. After some algebra the above equation can be rewritten as

$$\frac{1}{T} P_l(\omega) = \sum_{nn'} \int s^2 \mathrm{d}s\; A(s) F_l^2 U_{nl}(R_\odot) U_{n'l}(R_\odot) D_{nl}(s) D_{n'l}(s) L_{nl}(\omega) L_{n'l}^*(\omega) \tag{F.20}$$

where T is the time interval the spectrum is computed over, and

$$L_{nl}(\omega) = \frac{1}{2\pi} \frac{\gamma_{nl} - i\omega}{\omega_{nl}^2 + (\gamma_{nl} - i\omega)^2}\,. \tag{F.21}$$

Though we started from the certainly oversimplified assumption that the wave sources were spatially and temporally uncorrelated monopoles, equation (F.20) shows a number of important features. First of all, line asymmetry is seen as a result of the $n' \neq n$ terms. Also the effect of mode inertia is clear, from the presence of $U_{nl}(R_\odot) U_{n'l}(R_\odot)$.

F.4 Derivation of Kernels

We now return to the main task, the computation of travel-time kernels. The following definitions substantially simplify the upcoming algebra

$$a_l^i = \frac{2l+1}{4\pi} F_l^2 P_l(\cos\Delta_i)\,, \tag{F.22}$$

$$b_{nl} = U_{nl}^2(R_\odot) D_{nl}^2(s)\,, \tag{F.23}$$

$$c_{nl}(t) = \cos\omega_{nl} t \frac{\exp\left(-\gamma_{nl}|t|\right)}{4\gamma_{nl}}\,. \tag{F.24}$$

The index i on a_l^i denotes that a depends on the distance Δ_i. Neither b nor c depend on distance. In terms of the above definitions the zero-order cross-correlation (eq. [F.19]) is

$$C_i^0(t) = \sum_{nl} a_l^i b_{nl} c_{nl}(t)\,. \tag{F.25}$$

Here again the index i carries the distance dependence. As in section 3.2.2 we next perturb the zero order cross-correlation.

The perturbations to the eigenfunctions and frequencies due to a spherically symmetric perturbation $\delta\mathcal{L}$ to the wave operator $\mathcal{L}$ are (eq. [3.33])

$$\delta U_{nl}(R_\odot) = \sum_{n'}{}' \frac{\delta\mathcal{L}_{nn'}^{\,l}}{\omega_{nl}^2 - \omega_{n'l}^2} U_{n'l}(R_\odot)\,, \tag{F.26}$$

$$\delta D_{nl}(s) = \sum_{n'}{}' \frac{\delta\mathcal{L}_{nn'}^{\,l}}{\omega_{nl}^2 - \omega_{n'l}^2} D_{n'l}(s)\,, \tag{F.27}$$

$$\delta\omega_{nl} = \frac{\delta\mathcal{L}_{nn}^{\,l}}{2\omega_{nl}}\,. \tag{F.28}$$

The notation $\sum_{n'}'$ means the sum over n', excluding the term $n' = n$. The resulting perturbation to the cross-correlation is

$$\begin{aligned} \delta C_i(t) &= \sum_{nn'l} a_l^i D_{nl}(s) D_{n'l}(s) \delta\mathcal{L}_{nn'}^{\,l} \frac{U_{nl}^2 c_{nl}(t) - U_{n'l}^2(t) c_{n'l}(t)}{\omega_{nl}^2 - \omega_{n'l}^2} \\ &+ \sum_{nn'l} a_l^i U_{nl}(R_\odot) U_{n'l}(R_\odot) \delta\mathcal{L}_{nn'}^{\,l} \frac{D_{nl}^2 c_{nl}(t) - D_{n'l}^2 c_{n'l}(t)}{\omega_{nl}^2 - \omega_{n'l}^2} \\ &- \sum_{nl} a_l^i b_{nl} \delta\mathcal{L}_{nn}^{\,l} \frac{-t \sin(\omega_{nl} t)}{2\omega_{nl}} \frac{\exp(-\gamma_{nl} t)}{4\gamma_{nl}}\,. \end{aligned} \tag{F.29}$$

It is convenient to introduce

$$P = \int \mathrm{d}t\; \bar{C}^0(t) \ddot{C}^{\mathrm{ref}}(t - \tau^0)\,. \tag{F.30}$$

The following definitions will also be useful:

$$A^{i,l}_{nn'} = \frac{1}{P}\int \mathrm{d}t\ f(t-t_i)\dot{C}^{\mathrm{ref}}(t-\tau^0) \times \frac{U^2_{nl}(R_\odot)c_{nl}(t-t_i) - U^2_{n'l}(R_\odot)c_{nl}(t-t_i)}{\omega^2_{nl}-\omega^2_{n'l}}\,, \tag{F.31}$$

$$B^{i,l}_{nn'} = \frac{1}{P}\int \mathrm{d}t\ f(t-t_i)\dot{C}^{\mathrm{ref}}(t-\tau^0) \times \frac{D^2_{nl}(s)c_{nl}(t-t_i) - D^2_{n'l}(s)c_{nl}(t-t_i)}{\omega^2_{nl}-\omega^2_{n'l}}\,, \tag{F.32}$$

$$C^{i,l}_{nn'} = \delta_{nn'}\frac{1}{P}\int \mathrm{d}t\ f(t-t_i)\dot{C}^{\mathrm{ref}}(t-\tau^0) \times \frac{-(t-t_i)\sin\left[\omega_{nl}(t-t_i)\right]}{2\omega_{nl}}\frac{\exp\left(-\gamma_{nl}t\right)}{4\gamma_{nl}}\,. \tag{F.33}$$

Equation (F.10) can then be written, using equations (F.4) and (F.29), as

$$\delta\tau = \sum_{nn'l,i} a^i_l \delta\mathfrak{L}^l_{nn'}\left[D_{nl}(s)D_{n'l}(s)A^{i,l}_{nn'} + U_{nl}(R_\odot)U_{n'l}(R_\odot)B^{i,l}_{nn'} + b_{nl}C^{i,l}_{nn'}\right]\,. \tag{F.34}$$

For the case of sound speed perturbations

$$\delta\mathfrak{L}^l_{nn'} = \int_0^{R_\odot} \rho r^2 \mathrm{d}r\ \delta c^2 D_{nl}(r)D_{n'l}(r) \tag{F.35}$$

so

$$\delta\tau = \int_0^{R_\odot} \mathrm{d}r\ K(r)\frac{\delta c(r)}{c(r)} \tag{F.36}$$

with

$$K(r) = 2\rho c^2 r^2 \sum_{nn'l,i} a^i_l D_{nl}(r)D_{n'l}(r) \times \left[D_{nl}(s)D_{n'l}(s)A^{i,l}_{nn'} + U_{nl}(R_\odot)U_{n'l}(R_\odot)B^{i,l}_{nn'} + b_{nl}C^{i,l}_{nn'}\right]\,. \tag{F.37}$$

We used the above equation to compute the kernels for section 4.3.

Appendix G

Time-Domain Calculation of 3D Kernels

The purpose of this appendix is to describe some preliminary work that I have done on the time-domain computation of three-dimensional travel-time kernels for p modes. As we saw in section 3.4.1 the computation of travel-time kernels involves numerous time-domain integrals. We then saw in section 3.4.2 that with some reasonable assumptions regarding the source covariance matrix the computation could be written in a simple form in the Fourier domain.

In section 3.4.2 we looked at travel-time kernels for surface gravity waves, with a filter that only allowed waves with frequencies between two and four mHz. This filter ensured that we did not have to compute anything using waves with very small line-widths. For computations of p-mode kernels at large distances, we do have to take into account modes with very small damping rates (e.g. $n = 11$, $l = 0$ has a frequency of 1.7 mHz and a damping rate of 0.2 μHz). When these very weakly damped modes are present doing numerical integrations in the Fourier (temporal frequency) domain is difficult, as essentially the grid must be fine enough to resolve the lines in the power spectrum. Using a grid spacing of 0.02 μHz to cover a range of a few mHz gives a grid size of 10^5 grid points, which is doable, but not appealing.

Another approach would be to do a calculation in the time domain, working in the small damping approximation, as I do in Appendix F. In that particular case we were concerned only with kernels for sound speed perturbations that are constant on spheres and as a result did not have to worry about coupling between modes with different l, which vastly simplified the calculation. In the case of only coupling modes with same angular degree, there are no accidental degeneracies. In the general case, something must be done to treat modes that are accidentally degenerate in frequency.

The basic outline of my approach is as follows. I start with the basic formalism of section 3.4.1. I write the normal mode expansion of the Green's function, as in Appendix F. It is then straightforward, though painfully complicated, to write an expression for travel-time kernels, keeping all the calculations in the time domain. It is then a matter of doing algebra to obtain an expression for travel-time kernels in the weak-damping limit. As the algebra is quite complicated, these results have not been verified in detail, and may contain important mistakes. Rather than present the details here, I refer the interested reader to the web:
http://soi.stanford.edu/papers/dissertations/birch/time_domain_kernels.ps
for a research note detailing my progress on this problem.

Appendix H

Descriptions of Some Useful Codes

In this appendix I describe some of the MATLAB codes that I have developed for doing the work described in this dissertation. The list of codes here is quite far from complete. The codes described in this appendix, along with the supporting codes, can be found on the web (http://soi.Stanford.EDU/papers/dissertations/birch). The two sections of this appendix cover the codes for normal mode based calculations (section H.1) and ray theory calculations (section H.2).

H.1 Normal Mode Calculations

All of the routines in this section are based on a normal mode expansion of the Green's function (Dahlen & Tromp, 1998). In general these codes take as input a set of eigenfunctions, frequencies, and damping rates. This is described in detail in the on-line documentation. Most of the functions also require as estimate of the instrumental OTF.

- `run_single_xcorr.m`: This code computes the cross-correlation signal for a particular distance. The inputs are the normal-mode frequencies, damping rates, and the amplitudes with which each mode contributes to the cross-correlation. This amplitude factor can be computed from the filter function in the single-source model, or from the OTF and the source covariance matrix in the distributed-source model. This routine is one of the basic building blocks for many of the computations done in this dissertation. It can be used to do forward calculations for travel times, as the forward problem for normal-mode frequencies is already well known.

- run_kernel.m: This code computes slices through the 3D single-source kernels described in section 3.2 and can also compute 1D kernels.

H.2 Ray Calculations

This packages contains three main codes. There is one, run_tau.m, that computes first-skip distances and group times for rays with given frequency and angular degree. Another code, run_raypath.m, is available to compute full ray paths as well as the phase and group times along the ray path. Finally, ray_kernel.m uses the output of run_raypath.m to compute ray kernels for the effect of sound-speed perturbations on travel times. None of these codes do anything that is particularly new; they are all based on standard results from the literature. That said, however, I have found them very useful.

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