Acoustic scattering off flux tubes: Is the Born approximation valid?

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Abstract

With the aim of studying magnetic effects in time-distance helioseismology, we use the first-order Born approximation to compute the scattering of acoustic plane waves by a magnetic cylinder embedded in a uniform medium. We show, by comparison with the exact solution, that the travel-time shifts computed in the Born approximation are everywhere valid to first order in the ratio of the magnetic to the gas pressures. For arbitrary magnetic field strength, the Born approximation is not valid in the limit where the radius of the magnetic cylinder tends to zero.

Introduction

Time-distance helioseismology (Duvall et al., 1993) has been used to measure wave travel times in and around magnetic active regions and sunspots to estimate subsurface flows and wave-speed perturbations (e.g. Duvall et al., 1996). A challenging problem is to estimate the subsurface magnetic field from travel times. In order to do so, one must understand the dependence of the travel times on the magnetic field.

The interaction of acoustic waves with sunspot magnetic fields is quite strong in the near surface layers. As a result, the effect of the magnetic field on the travel times is not expected to be small near the surface. Deeper inside the Sun, however, the ratio of the magnetic pressure to the gas pressure becomes small, and it is tempting to treat the effects of the magnetic field on the waves using perturbation theory. Of particular interest is the search for a magnetic field at the bottom of the convection zone. Such a linear inversion scheme has been proposed by Kosovichev & Duvall (1997) for time-distance helioseismology using the ray approximation, but it needs to be extended to finite wavelengths.

The problem

We start with the ideal equations of magnetohydrodynamics. We solve the equations of continuity, momentum, magnetic induction, Gauss' law for the magnetic field and utilize the ideal gas law (e.g. Gizon, Hanasoge & Birch, 2006). We denote density by ρ, velocity by $\boldsymbol{v}$, pressure by p, temperature by T and the magnetic field by $\boldsymbol{B}$. We consider a magnetic cylinder with radius R and uniform magnetic field strength B_t embedded in an infinite, otherwise uniform, gravity free medium with constant density ρ_0, gas pressure p_0, and temperature T_0. We use a cylindrical coordinate system (r, θ, z) where r is the radial coordinate, θ is the azimuthal angle, and z is the vertical coordinate in the direction of the cylinder axis. We denote the corresponding unit vectors by $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$. All steady physical quantities are denoted with an overbar. In particular, we have

$$\overline{\boldsymbol{B}} = B_t \Theta(R-r)\hat{\boldsymbol{z}}, \quad (1)$$
$$\overline{\rho} = \rho_t \Theta(R-r) + \rho_0 \Theta(r-R), \quad (2)$$
$$\overline{p} = p_t \Theta(R-r) + p_0 \Theta(r-R), \quad (3)$$

where the Heaviside step function is defined by $\Theta(r) = 0$ if $r < 0$ and $\Theta(r) = 1$ if $r > 0$.

The density and pressure inside the tube are ρ_t and p_t respectively. We assume that there is no mean flow in this problem, i.e. $\overline{\boldsymbol{v}} = 0$. We choose to study the case where the background temperature is the same inside and outside the magnetized region, resulting in constant sound speed everyewhere.

Linear Waves

In this calculation, we only study linear waves on a steady background. The magnetic field $\overline{\boldsymbol{B}}$ and all other background quantities do not depend on z. Thus, a wave with a z dependence of the form $e^{ik_z z}$ will have the same z dependence after interacting with the magnetic cylinder. As a result, we study solutions where the pressure fluctuations are of the form

$$p'(\boldsymbol{r}, z, t) = \tilde{p}(\boldsymbol{r}) \exp(ik_z z - i\omega t), \quad (4)$$

where $\boldsymbol{r} = (r, \theta)$ is a position vector perpendicular to the tube axis. All the other wave variables, ρ', $\boldsymbol{v}'$, and $\boldsymbol{B}'$ are written in the same form as equation (4). A plane wave can be expanded in cylindrical coordinates as a sum over azimuthal components (index m) according to (e.g. Bogdan, 1989):

$$\tilde{p}_{\rm inc}(\boldsymbol{r}) = P \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{im\phi}, \quad (5)$$

where J_m denotes the Bessel function of order m and ϕ is the angle between $\boldsymbol{k}$ and $\boldsymbol{r}$. Applying the boundary condition that the total wave pressure, hydrodynamic plus magnetic, and the radial velocity must be continuous across the tube boundary, the *exact* linear (small amplitude waves) scattered wavefield due to the incident wave, given by equation (5), may be computed (Wilson, 1980):

$$\tilde{p}(\boldsymbol{r}) = \begin{cases} P \sum_m i^m B_m J_m(k_t r) e^{im\phi} & (r < R) \\ \tilde{p}_{\rm inc} + P \sum_m i^m A_m H_m(kr) e^{im\phi} & (r > R) \end{cases} \quad (6)$$

where $H_m = H_m^{(1)}$ is the Hankel function of the first kind of order m. The quantity k_t is the horizontal wavenumber inside the tube.

First Born Approximation

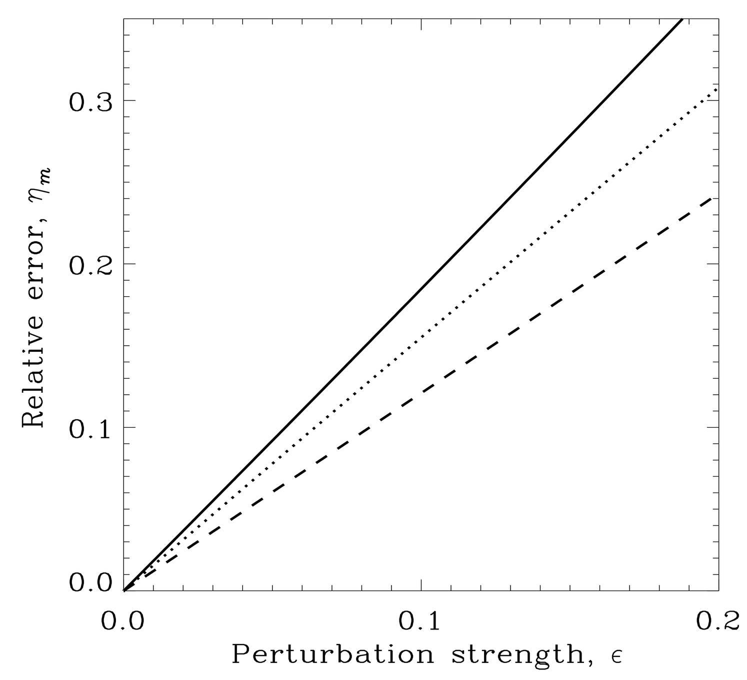
Details regarding the application of the Born approximation to the problem at hand may be found in Gizon, Hanasoge & Birch (2006) and references therein. Suffice it to say that after considerable mathematical manipulation, we are able to obtain an expression for the scattered wave-field akin to equation (6):

$$p_{Born}(\boldsymbol{r}) = p_{inc} + P \sum_{m=-\infty}^{\infty} i^m e^{im\phi} \times \begin{cases} C_m J_m(kr) - \epsilon \frac{kr}{2} J'_m(kr) & r < R \\ A_m^{\rm Born} H_m(kr) & r > R, \end{cases} \quad (7)$$

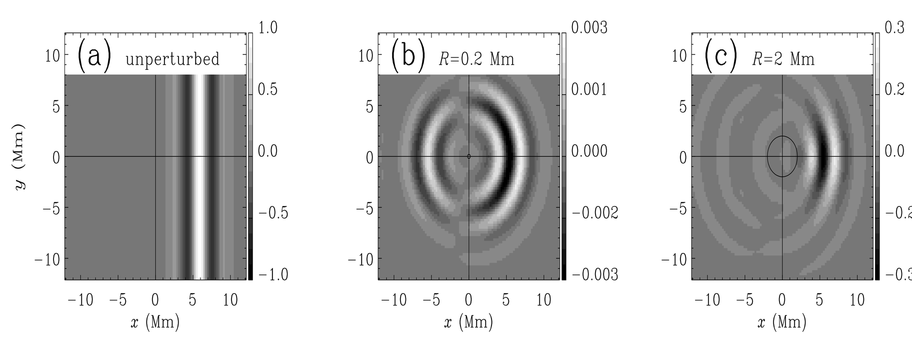
where the coefficients $A_m^{\rm Born}$ and C_m are described in Appendix C Gizon, Hanasoge & Birch (2006). We are also able to show analytically that the Born approximation approaches the exact solution in the limit of small ϵ, where ϵ is a non-dimensional parameter defined as:

$$\epsilon = \frac{B_t^2}{4\pi\rho_0 c^2}. \quad (8)$$

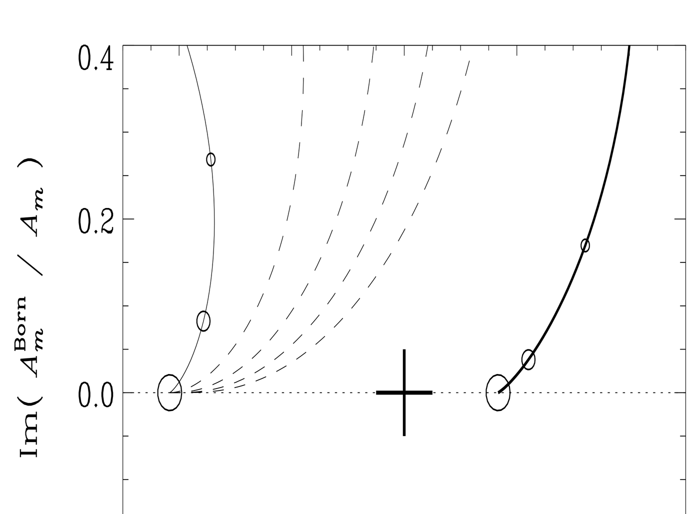
Denoting the coefficients A_m (from equation (6)) obtained in the Born limit by $A_m^{\rm Born}$, we have the following comparative results.



Plotted in this figure is the fractional error $\eta_m = |A_m^{\rm Born} - A_m|/|A_m|$ as a function of ϵ for $m = 0$ (solid line), $m = 1$ (dotted line), and $m = 2$ (dashed line) for the case $R = 2$ Mm, $\omega/2\pi = 3$ mHz, and $k_z = 0$.



Plots of the pressure field of the unperturbed (a) and scattered (b - c) wavepackets at time $t = 8.9$ min after the unperturbed wavepacket has crossed the center of the magnetic cylinder. The wavepacket parameters are described in the text. The wavevector is normal to the axis of the magnetic cylinder and in the $+\hat{\mathbf{x}}$ direction. The strength of the magnetic perturbation is $\epsilon = 0.13$. In panel (b) the tube radius is $R = 0.2$ Mm and in panel (c) is the tube radius is $R = 2$ Mm. The circles outline the cross-section of the tube. Notice that the gray scales are different in each panel. The backscattered wave is more prominent for the tube with the smaller radius (panel b).



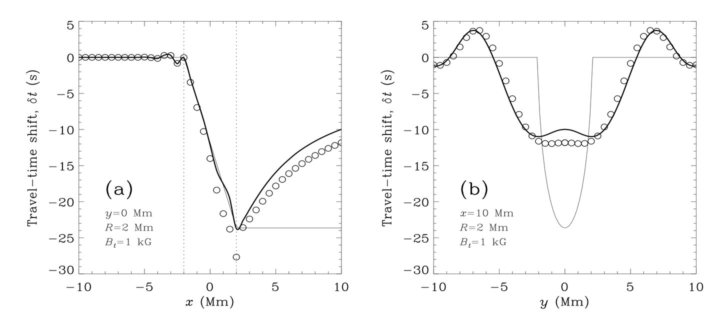
Ratio $A_m^{\rm Born}/A_m$ in the complex plane at fixed $\epsilon = 1$ and $k_z = 0$. The ratio is plotted for varying values of the tube radius in the cases $m = 0$ (thick line), $m = 1$ (thin line), and $2 \le m \le 5$ (dashed lines). The big circles show the limit $kR \to 0$. If the Born approximation were correct for small tube radii, the big circles would coincide with the cross. The small and medium-size circles are for $kR = 1$ and $kR = 1/2$ respectively.

Traveltimes

We define the travel-time shifts caused by the magnetic cylinder as the time $\delta t(\boldsymbol{r})$ which minimizes the function

$$X(t) = \int dt' \left[p'(\boldsymbol{r}, t') - p'_{\rm inc}(\boldsymbol{r}, t' - t)\right]^2, \quad (9)$$

where p' is the full wavefield that includes both the incident wavepacket and the scattered wave packet caused by the magnetic field. The travel-time shifts can be computed in this way for either the exact solution or the Born-approximation. In addition, it is also interesting to compare with the ray approximation as given by equation (14) from Kosovichev & Duvall (1997).



Local travel-time shifts $\delta t(\boldsymbol{r})$ caused by the magnetic cylinder ($\epsilon = 0.13$). The travel times are measured at positions $\boldsymbol{r}$ in a plane perpendicular to both the cylinder axis. The incoming wavepacket moves in the $+\hat{\mathbf{x}}$ direction. The radius of the tube is $R = 2$ Mm and the tube axis is $(x, y) = (0, 0)$. In both panels the heavy solid line is the exact travel-time shifts, the circles are the Born travel-time shifts, and the light line gives the ray approximation. The left panel shows the travel-time shifts as a function of x at fixed $y = 0$. The right panel shows the travel-time shifts as a function of y at fixed $x = 10$ Mm. The Born approximation is reasonable for this value of ϵ. The ray-approximation does not capture finite-wavelength effects and fails to describe wavefront healing (Nolet & Dahlen, 2000).

Conclusions

The sensitivity of travel times to local perturbations in internal solar properties can be described through linear sensitivity functions, also called travel-time kernels. The present work suggests that travel-time kernels for the subsurface magnetic field will be useful for probing depths greater than a few hundred km beneath the photosphere, at least in the case when the travel times are measured between surface points that are not in magnetic regions.

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