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1+1+2 gravitational perturbations on LRS class II spacetimes: decoupling gravito-electromagnetic tensor harmonic amplitudes

R B Burston

Max Planck Institute for Solar System Research, 37191 Katlenburg-Lindau, Germany

E-mail: burston@mps.mpg.de

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Abstract

This is the first in a series of papers which considers gauge-invariant and covariant gravitational perturbations on arbitrary vacuum locally rotationally symmetric (LRS) class II spacetimes. Ultimately, we derive four decoupled equations governing four specific combinations of the gravito-electromagnetic (GEM) 2-tensor harmonic amplitudes. We use the gauge-invariant and covariant 1+1+2 formalism which Clarkson and Barrett (2003 *Class. Quantum Grav.* **20** 3855) developed for analysis of vacuum Schwarzschild perturbations. In particular we focus on the first-order 1+1+2 GEM system and use linear algebra techniques suitable for exploiting its structure. Consequently, we express the GEM system new 1+1+2 complex form by choosing new complex GEM tensors, which is conducive to decoupling. We then show how to derive a gauge-invariant and covariant decoupled equation governing a newly defined complex GEM 2-tensor. Finally, the GEM 2-tensor is expanded in terms of arbitrary tensor harmonics and linear algebra is used once again to decouple the system further into four real decoupled equations.

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1. Introduction

The gauge-invariant and covariant 1+1+2 formalism was first developed by Clarkson and Barrett [1] for an analysis of vacuum gravitational perturbations to a covariant Schwarzschild spacetime. This was further developed in [2], who considered both scalar and electromagnetic (EM) perturbations to arbitrary locally rotationally symmetric (LRS) class II spacetimes [3–5], where they were able to derive generalized Regge–Wheeler [6] (RW) equations governing the 1+1+2 EM scalars, $\mathscr{E}$ and $\mathscr{B}$. Subsequent to this, we also considered EM perturbations to LRS class II spacetimes [7, 8]. Therein, we used linear algebra techniques to show that the first-order 1+1+2 Maxwell's equations naturally decouple by choosing new complex variables. Consequently, we expressed Maxwell's equations in a new 1+1+2 complex form that is suited to decoupling. We reproduced the generalized RW result in a new complex form and further established that the EM 2-vectors, $\mathscr{E}_\mu$ and $\mathscr{B}_\mu$, also decouple from the EM scalars. The EM 2-vectors were expanded into two polar perturbations $\{\mathscr{E}_\mathrm{V}, \bar{\mathscr{B}}_\mathrm{V}\}$ and two axial perturbations $\{\bar{\mathscr{E}}_\mathrm{V}, \mathscr{B}_\mathrm{V}\}$ using the arbitrary vector harmonic expansion developed in [1, 2]. Finally, we once again used linear algebra techniques, and derived four real decoupled equations governing the four combinations of the 2-vector harmonic amplitudes [8]. The precise combinations which decoupled were found to be, for the polar perturbations $\{\mathscr{E}_\mathrm{V} - \bar{\mathscr{B}}_\mathrm{V}, \mathscr{E}_\mathrm{V} + \bar{\mathscr{B}}_\mathrm{V}\}$, and for the axial perturbations $\{\bar{\mathscr{E}}_\mathrm{V} - \mathscr{B}_\mathrm{V}, \bar{\mathscr{E}}_\mathrm{V} + \mathscr{B}_\mathrm{V}\}$.

In this paper, we consider both gravitational and energy–momentum perturbations to arbitrary vacuum LRS class II spacetimes using the 1+1+2 formalism. The primary focus is with the first-order GEM system as it is well established to have remarkably similar mathematical structure to Maxwell's equations [9, 10]. We use similar techniques as in [8], which was successful in fully decoupling the EM 2-vector harmonic components, and ultimately show that this is also successful for fully decoupling the GEM 2-tensor harmonic components.

In section 2, we collate the important results arising from Clarkson and Barrett's 1+1+2 formalism and the background LRS class II spacetime is reproduced from [2]. Also, the scalar and 2-vector harmonic expansion formalism is taken from [2], and we provide a new generalization of the spherical tensor harmonics developed in [1] for tensor harmonics. We use precisely the same notation as in [1, 2, 8] as well as introducing some new quantities which are well defined throughout. In section 3, we carefully define the first-order perturbations (including the energy–momentum quantities) to be gauge-invariant according to the Sachs–Stewart–Walker lemma [11, 12]. We proceed to write the first-order GEM system, conservation equations and the Ricci identities. In section 5 we derive the decoupled equations and consider tensor harmonic expansions.

2. Preliminaries

The purpose of this section is to present the necessary results for the current series of papers on gravitational and energy–momentum perturbations to arbitrary vacuum LRS class II spacetimes.

2.1. Clarkson and Barrett's 1+1+2 formalism

The 1+3 formalism is very well-established (see, for example, [3, 9, 13]) whereby, a 4-velocity u^μ is defined such that it is both time-like and normalized ($u^\alpha u_\alpha = -1$). Consequently, all quantities and governing equations are decomposed by projecting onto a 3-sheet which is orthogonal to u^μ, and hence they are called 3-tensors, and in the time-like direction. The essential ingredient for Clarkson and Barrett's 1+1+2 formalism is to further decompose the 1+3 formalism by introducing a new 'radial' vector n^μ which is space-like, normalized ($n^\alpha n_\alpha = 1$) and orthogonal to u^μ. In this way, all 3-tensors may be further decomposed into 2-tensors which have been projected onto the 2-sheet orthogonal to both n^μ and u^μ and in the radial direction. The covariant derivative of the 4-velocity in standard 1+3 notation is

$$\nabla_\mu u_\nu = \sigma_{\mu\nu} + \tfrac{1}{3}\theta h_{\mu\nu} - u_\mu \dot{u}_\nu + \epsilon_{\mu\nu\alpha}\omega^\alpha, \tag{1}$$

where ∇_μ is the covariant derivative operator, $\sigma_{\mu\nu}$ and θ are the shear and expansion of the 3-sheets, $h_{\mu\nu}$ is a tensor that projects onto the 3-sheets, $\epsilon_{\mu\nu\sigma}$ is the Levi-Civita 3-tensor and

ω^μ is the vorticity. Finally, the acceleration vector is $\dot{u}^\mu$ where the 'dot' derivative is defined as $\dot{X}_{\mu\ldots\nu} := u^\alpha \nabla_\alpha X_{\mu\ldots\nu}$ and $X_{\mu\ldots\nu}$ represents any quantity. Clarkson and Barrett irreducibly split these standard 1+3 quantities into 1+1+2 form according to

$$\dot{u}_\mu = \mathcal{A} n_\mu + \mathcal{A}_\mu, \tag{2}$$

$$\omega_\mu = \Omega n_\mu + \Omega_\mu, \tag{3}$$

$$\sigma_{\mu\nu} = \Sigma_{\mu\nu} - \tfrac{1}{2} N_{\mu\nu} \Sigma + 2\Sigma_{(\mu} n_{\nu)} + \Sigma n_\mu n_\nu. \tag{4}$$

The 1+3 GEM fields are also decomposed,

$$E_{\mu\nu} = \mathcal{E}_{\mu\nu} - \tfrac{1}{2} N_{\mu\nu} \mathcal{E} + 2\mathcal{E}_{(\mu} n_{\nu)} + \mathcal{E} n_\mu n_\nu, \tag{5}$$

$$H_{\mu\nu} = \mathcal{H}_{\mu\nu} - \tfrac{1}{2} N_{\mu\nu} \mathcal{H} + 2\mathcal{H}_{(\mu} n_{\nu)} + \mathcal{H} n_\mu n_\nu, \tag{6}$$

where $E_{\mu\nu}$ and $H_{\mu\nu}$ are respectively the electric and magnetic parts of the Weyl tensor, $C_{\mu\nu\sigma\tau}$. In a similar fashion, the 3-covariant derivative (D_μ) of the radial vector is decomposed into 1+1+2 form according to

$$D_\mu n_\nu = n_\mu a_\nu + \tfrac{1}{2}\phi N_{\mu\nu} + \xi \epsilon_{\mu\nu} + \zeta_{\mu\nu}, \tag{7}$$

where $\zeta_{\mu\nu}$ and ϕ are respectively the shear and expansion of the 2-sheets, $N_{\mu\nu}$ is a tensor that projects onto the 2-sheets, ξ represents the twisting of the sheet and $\epsilon_{\mu\nu}$ is the Levi-Civita 2-tensor. Also, the acceleration 2-vector is $a_\mu := \hat{n}_\mu$ where the 'hat' derivative is defined as $\hat{W}_{\mu\ldots\nu} := n^\alpha D_\alpha W_{\mu\ldots\nu}$ and $W_{\mu\ldots\nu}$ represents a 3-tensor. Finally, the 'dot' derivative of the radial normal is also split according to

$$\dot{n}_\mu = \mathcal{A} u_\mu + \alpha_\mu. \tag{8}$$

Therefore, the irreducible set of 1+1+2 quantities, and in accord with standard terminology, is

$$\begin{aligned} &\text{scalars:} && \{\mathcal{A}, \phi, \Sigma, \theta, \mathcal{E}, \mathcal{H}, \Lambda, \xi, \Omega\}, \\ &\text{2-vectors:} && \{a^\mu, \alpha^\mu, \Omega^\mu, \mathcal{A}^\mu, \Sigma^\mu, \mathcal{E}^\mu, \mathcal{H}^\mu\}, \\ &\text{2-tensors:} && \{\Sigma_{\mu\nu}, \zeta_{\mu\nu}, \mathcal{E}_{\mu\nu}, \mathcal{H}_{\mu\nu}\}, \end{aligned} \tag{9}$$

where the cosmological constant (Λ) has also been included. Furthermore, the energy–momentum quantities, heat-flux and anisotropic pressure, become respectively [2],

$$q^\mu = \mathcal{Q} n^\mu + \mathcal{Q}^\mu, \tag{10}$$

$$\pi_{\mu\nu} = \Pi_{\mu\nu} - \tfrac{1}{2} N_{\mu\nu} \Pi + 2\Pi_{(\mu} n_{\nu)} + \Pi n_\mu n_\nu. \tag{11}$$

Thus, the irreducible 1+1+2 energy–momentum quantities are

$$\text{scalars: } \{\mu, p, \mathcal{Q}, \Pi\}, \qquad \text{2-vectors: } \{\mathcal{Q}^\mu, \Pi^\mu\} \qquad \text{and} \qquad \text{2-tensor: } \{\Pi_{\mu\nu}\}. \tag{12}$$

where μ is the mass–energy density and p is the isotropic pressure.

2.2. Background vacuum LRS class II spacetime

The background comprises the most general vacuum LRS class II spacetime and is defined by six non-vanishing LRS class II scalars

$$\text{LRS class II:} \quad \{\mathcal{A}, \phi, \Sigma, \theta, \mathcal{E}, \Lambda\}. \tag{13}$$

Popular examples of LRS class II backgrounds include the Schwarzschild spacetime as presented in [1] using a static coordinate system (diagonal metric) where $(\mathcal{A}, \phi, \mathcal{E}) \neq 0$

and $\Sigma = \theta = \Lambda = 0$. Furthermore, any other coordinate system (for example, a freely falling observer) describing the Schwarzschild spacetime may also be implemented but the LRS class II scalars will change for each case.

The background Ricci identities for both u^μ and n^μ and the Bianchi identities yield a set of evolution and propagation equations governing these scalars. They were first presented in [1] for a covariant Schwarzschild spacetime and generalized to non-vacuum LRS class II spacetimes in [2] for which we reproduce them here for the vacuum case,

$$\left(\mathcal{L}_n + \tfrac{1}{2}\phi\right)\phi + \left(\Sigma - \tfrac{2}{3}\theta\right)\left(\Sigma + \tfrac{1}{3}\theta\right) + \mathcal{E} + \tfrac{2}{3}\Lambda = 0, \tag{14}$$

$$\left(\mathcal{L}_n + \tfrac{3}{2}\phi\right)\Sigma - \tfrac{2}{3}\mathcal{L}_n\theta = 0, \tag{15}$$

$$\left(\mathcal{L}_n + \tfrac{3}{2}\phi\right)\mathcal{E} = 0, \tag{16}$$

$$\left(\mathcal{L}_u - \tfrac{1}{2}\Sigma + \tfrac{1}{3}\theta\right)\phi + \mathcal{A}\left(\Sigma - \tfrac{2}{3}\theta\right) = 0, \tag{17}$$

$$\left(\mathcal{L}_u - \tfrac{1}{2}\Sigma + \tfrac{1}{3}\theta\right)\left(\Sigma - \tfrac{2}{3}\theta\right) + \mathcal{A}\phi + \mathcal{E} + \tfrac{2}{3}\Lambda = 0, \tag{18}$$

$$\left(\mathcal{L}_u - \tfrac{3}{2}\Sigma + \theta\right)\mathcal{E} = 0, \tag{19}$$

$$\left(\mathcal{L}_n + \mathcal{A} - \tfrac{1}{2}\phi\right)\mathcal{A} - \tfrac{3}{2}\left(\mathcal{L}_u + \tfrac{1}{2}\Sigma + \tfrac{2}{3}\theta\right)\Sigma - \tfrac{3}{2}\mathcal{E} + \Lambda = 0, \tag{20}$$

$$\left(\mathcal{L}_u + \Sigma + \tfrac{1}{3}\theta\right)\left(\Sigma + \tfrac{1}{3}\theta\right) - (\mathcal{L}_n + \mathcal{A})\mathcal{A} + \mathcal{E}, \tag{21}$$

$$\delta_\mu\mathcal{E} = \delta_\mu\phi = \delta_\mu\mathcal{A} = \delta_\mu\theta = \delta_\mu\Sigma = 0, \tag{22}$$

where δ_μ is the covariant 2-derivative associated with the 2-sheet. Moreover, in addition to the 'dot' and 'hat' derivatives, we will also use the Lie derivative, $\mathcal{L}_u$ and $\mathcal{L}_n$ (where, for example, the standard definition can be found in [14]). This allows us to neatly express the equations using covariant differential operators. Since the system (14)–(22) considers only scalars, they simply become usual directional derivatives in this case and are equivalent to the 'dot' and 'hat' derivatives,

$$\mathcal{L}_u\psi = \dot{\psi} \qquad \text{and} \qquad \mathcal{L}_n\psi = \hat{\psi}. \tag{23}$$

Furthermore, for a 2-vector ψ_μ and 2-tensor $\psi_{\mu\nu}$, they are related as follows,

$$\left(\mathcal{L}_n - \tfrac{1}{2}\phi\right)\psi_{\bar{\mu}} = \hat{\psi}_{\bar{\mu}} \qquad \text{and} \qquad (\mathcal{L}_n - \phi)\psi_{\bar{\mu}\bar{\nu}} = \hat{\psi}_{\bar{\mu}\bar{\nu}}, \tag{24}$$

$$\left(\mathcal{L}_u + \tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta\right)\psi_{\bar{\mu}} = \dot{\psi}_{\bar{\mu}} \qquad \text{and} \qquad \left(\mathcal{L}_u + \Sigma - \tfrac{2}{3}\theta\right)\psi_{\bar{\mu}\bar{\nu}} = \dot{\psi}_{\bar{\mu}\bar{\nu}}. \tag{25}$$

It is also convenient to introduce five more definitions for the 2-gradients of the LRS class II scalars that arise in (22). Three of these arise in [1],

$$X_\mu := \delta_\mu\mathcal{E}, \qquad Y_\mu := \delta_\mu\phi \qquad \text{and} \qquad Z_\mu := \delta_\mu\mathcal{A}, \tag{26}$$

and two new definitions are made to account for the additional complications of an arbitrary LRS class II background,

$$V_\mu := \delta_\mu\left(\Sigma + \tfrac{1}{3}\theta\right) \qquad \text{and} \qquad W_\mu := \delta_\mu\left(\Sigma - \tfrac{2}{3}\theta\right). \tag{27}$$

Finally, as in [2] we also find it useful to work with the extrinsic curvature and it also comes with evolution and propagation equations,

$$K = \tfrac{1}{4}\phi^2 - \tfrac{1}{4}\left(\Sigma - \tfrac{2}{3}\theta\right)^2 - \mathcal{E} + \tfrac{1}{3}\Lambda, \tag{28}$$

$$(\mathcal{L}_n + \phi)K = 0 \qquad \text{and} \qquad \left(\mathcal{L}_u - \Sigma + \tfrac{2}{3}\theta\right)K = 0. \tag{29}$$

2.3. Harmonic expansions

The approach to perturbation problems by expanding first-order quantities into harmonic components is very common and a good review on spherical harmonics is presented in [15]. Harmonic expansions were also used by [16, 17], defined according to [18–20], who analyzed both EM and gravitational perturbations of charged black holes in n dimensions. They showed that only specific combinations of the first-order EM and gravitational quantities satisfy decoupled equations, as was demonstrated for the traditional four-dimensional case [21, 22]. The spherical harmonic expansions for 1+1+2 scalars, 2-vectors and 2-tensors were first presented in [1] for the specific Schwarzschild case. This was subsequently generalized to harmonic expansions for both scalars and 2-vectors in [2]. In this section, we reproduce the necessary results from [2] as well as include a new generalization of the 2-tensor spherical harmonics in [1] to 2-tensor harmonics. Dimensionless sheet harmonic functions Q (defined on the background) are defined as

$$\delta^2 Q := -\frac{k^2}{r^2} Q \qquad \text{and} \qquad \hat{Q} = \dot{Q} = 0, \tag{30}$$

where k^2 is real and the 2-Laplacian is defined as $\delta^2 := \delta^\alpha \delta_\alpha$. The scalar function r is defined by the following covariant equations:

$$\left(\mathcal{L}_n - \tfrac{1}{2}\phi\right) r = 0, \qquad \left(\mathcal{L}_u + \tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta\right) r = 0 \qquad \text{and} \qquad \delta_\mu r = 0. \tag{31}$$

Now any first-order scalar function can be expanded as

$$\psi = \sum_k \psi_{\mathrm{S}}^{(k)} Q^{(k)} = \psi_{\mathrm{S}} Q, \tag{32}$$

where ψ_{S} is the scalar harmonic amplitude and the summation over k is implicit in the last equality. Similarly, all vectors are expanded in terms of even (Q_μ) and odd ($\bar{Q}_\mu$) parity vector harmonics which are defined respectively,

$$Q_\mu = r \delta_\mu Q \qquad \to \qquad \delta^2 Q_\mu = \left(K - \frac{k^2}{r^2}\right) Q_\mu, \tag{33}$$

$$\bar{Q}_\mu = r \epsilon_\mu{}^\alpha \delta_\alpha Q \qquad \to \qquad \delta^2 \bar{Q}_\mu = \left(K - \frac{k^2}{r^2}\right) \bar{Q}_\mu. \tag{34}$$

The vector harmonics are orthogonal ($Q^\alpha \bar{Q}_\alpha = 0$) and they have the following properties: $\bar{Q}_\mu = \epsilon_\mu{}^\alpha Q_\alpha$ and $Q_\mu = -\epsilon_\mu{}^\alpha \bar{Q}_\alpha$. Thus any first-order vector may be expanded according to

$$\psi_\mu = \sum_k \psi_{\mathrm{V}}^{(k)} Q_\mu^{(k)} + \bar{\psi}_{\mathrm{V}}^{(k)} \bar{Q}_\mu^{(k)} = \psi_{\mathrm{V}} Q_\mu + \bar{\psi}_{\mathrm{V}} \bar{Q}_\mu, \tag{35}$$

where similarly ψ_{V} and $\bar{\psi}_{\mathrm{V}}$ are the vector harmonic amplitudes and the summation in the last quantity is implicit. Also note that the 2-Laplacian acting on the vector harmonics in (33)–(34) is written in terms of the Gaussian curvature here, whereas in [2] they use a further constraint of $K = 1/r^2$ which amounts to choosing a particular normalization that was convenient for their analysis.

We now present a generalization of the spherical tensor harmonics presented in [1] to tensor harmonics in arbitrary LRS class II spacetimes. The even and odd tensor harmonics are defined respectively,

$$Q_{\mu\nu} = r^2 \delta_{\{\mu} \delta_{\nu\}} Q, \qquad \delta^2 Q_{\mu\nu} = \left(4K - \frac{k^2}{r^2}\right) Q_{\mu\nu}, \tag{36}$$

$$\bar{Q}_{\mu\nu} = r^2 \epsilon_{\alpha\{\mu} \delta^\alpha \delta_{\nu\}} Q, \qquad \delta^2 \bar{Q}_{\mu\nu} = \left(4K - \frac{k^2}{r^2}\right) \bar{Q}_{\mu\nu}, \tag{37}$$

where the 'curly' brackets indicate the part that is symmetric and trace-free with respect to the 2-sheet. These are orthogonal ($Q^{\alpha\beta}\bar{Q}_{\alpha\beta} = 0$) and have the following properties: $Q_{\mu\nu} = \epsilon_{(\mu}{}^{\alpha}\bar{Q}_{\nu)\alpha}$ and $\bar{Q}_{\mu\nu} = -\epsilon_{(\mu}{}^{\alpha}Q_{\nu)\alpha}$. Therefore, all first-order tensors may now be expanded in terms of tensor harmonics according to

$$\psi_{\mu\nu} = \sum_k \psi_{\mathrm{T}}^{(k)} Q_{\mu\nu}^{(k)} + \bar{\psi}_{\mathrm{T}}^{(k)} \bar{Q}_{\mu\nu}^{(k)} = \psi_{\mathrm{T}} Q_{\mu\nu} + \bar{\psi}_{\mathrm{T}} \bar{Q}_{\mu\nu}, \tag{38}$$

where in accord with usual terminology, ψ_{T} and $\bar{\psi}_{\mathrm{T}}$ are the tensor harmonic amplitudes and again the summation in the last equality is implicit. We also have the following relationships which also generalize those presented in [1],

$$\delta^{\alpha}\psi_{\mu\alpha} = \frac{r}{2}\left(2K - \frac{k^2}{r^2}\right)(\psi_{\mathrm{T}} Q_{\mu} - \bar{\psi}_{\mathrm{T}} \bar{Q}_{\mu}), \tag{39}$$

$$\epsilon_{\{\mu}{}^{\alpha}\delta^{\beta}\psi_{\beta\}\alpha} = \frac{r}{2}\left(2K - \frac{k^2}{r^2}\right)(\bar{\psi}_{\mathrm{T}} Q_{\mu} + \psi_{\mathrm{T}} \bar{Q}_{\mu}). \tag{40}$$

3. The gravitational and energy–momentum perturbations

We now consider both gravitational and energy–momentum perturbations to the background LRS class II spacetime defined in section 2.2. In agreement with traditional practice we let all gravitational and energy–momentum quantities that vanish on the background LRS class II spacetime simply become quantities of first order (ϵ), i.e.

$$\text{first-order scalars:} \quad \{\mathcal{H}, \xi, \Omega, \mu, p, \mathcal{Q}, \Pi\} = \mathcal{O}(\epsilon), \tag{41}$$

$$\text{first-order 2-vectors:} \quad \{a^{\mu}, \alpha^{\mu}, \Omega^{\mu}, \mathcal{A}^{\mu}, \Sigma^{\mu}, \mathcal{E}^{\mu}, \mathcal{H}^{\mu}, \mathcal{Q}^{\mu}, \Pi^{\mu}\} = \mathcal{O}(\epsilon), \tag{42}$$

$$\text{first-order 2-tensors:} \quad \{\Sigma_{\mu\nu}, \zeta_{\mu\nu}, \mathcal{E}_{\mu\nu}, \mathcal{H}_{\mu\nu}, \Pi_{\mu\nu}\} = \mathcal{O}(\epsilon). \tag{43}$$

The first-order quantities given in (41)–(43) are all gauge-invariant under infinitesimal coordinate transformations, or more formally due to the Sachs–Stewart–Walker lemma [11, 12], as their corresponding background terms vanish. Furthermore, there is also the issue of choosing a particular frame in the perturbed spacetime (i.e. choosing the first-order 4-velocity and radial vector) as also discussed in [1]. In general, the first-order gauge-invariant 1+1+2 quantities will not be frame invariant as they naturally depend on this choice since their underlying definitions are typically just projections and contractions with the 4-velocity and radial vector.

Now consider some perturbed quantity, $\tilde{\psi}$, this is expanded to first-order according to

$$\tilde{\psi} = \psi + \delta\psi, \tag{44}$$

where ψ is the corresponding background value and $\delta\psi$ is the corresponding first-order part (and δ is not to be confused with the covariant 2-derivative δ_{μ}). Therefore, there are five LRS class II scalars which do not vanish on the background, and thus, they will experience first-order increments given by

$$\{\delta\mathcal{A}, \delta\phi, \delta\Sigma, \delta\theta, \delta\mathcal{E}\} = \mathcal{O}(\epsilon). \tag{45}$$

Furthermore, these five first-order scalars (45) are not gauge-invariant under the Sachs–Stewart–Walker lemma. However, as initiated in [1], the 2-gradient of these scalars does vanish on the background according to (22) and therefore, they become gauge-invariant quantities of first order,

$$\text{first-order 2-vectors:} \quad \{V_{\mu}, W_{\mu}, X_{\mu}, Y_{\mu}, Z_{\mu}\} = \mathcal{O}(\epsilon). \tag{46}$$

Throughout the remainder of this paper, every equation is written in a purely gauge-invariant way. This is predominately achieved by writing everything explicitly in terms of the quantities defined in (41)–(42) and (46), otherwise, it is ensured that particular combinations of gauge-variant quantities are written as one combined gauge-invariant quantity.

4. The first-order Bianchi and Ricci identities

The equations governing the first-order gauge-invariant 1+1+2 variables are found by decomposing the Ricci identities for both u^μ and n^μ, the once contracted Bianchi identities (GEM system) and the twice contracted Bianchi identities.

4.1. Twice-contracted Bianchi identities

In this paper we consider the first-order energy–momentum quantities as a known source that is capable of physically perturbing the background spacetime giving rise to first-order gravitational fields. Therefore, we begin with the conservation of mass equations as they will indicate how these first-order energy–momentum quantities propagate and evolve[1],

$$(\mathcal{L}_u + \theta)\mu + (\mathcal{L}_n + 2\mathcal{A} + \phi)\mathcal{Q} + \delta^\alpha \mathcal{Q}_\alpha + p\theta + \tfrac{3}{2}\Pi\Sigma = 0, \tag{47}$$

$$\left(\mathcal{L}_u + \Sigma + \tfrac{4}{3}\theta\right)\mathcal{Q} + (\mathcal{L}_n + \mathcal{A})p + \mu\mathcal{A} + \delta^\alpha \Pi_\alpha + \left(\mathcal{L}_n + \mathcal{A} + \tfrac{3}{2}\phi\right)\Pi = 0, \tag{48}$$

$$(\mathcal{L}_u + \theta)\mathcal{Q}_{\bar{\mu}} + (\mathcal{L}_n + \mathcal{A} + \phi)\Pi_{\bar{\mu}} + \delta_\mu\left(p - \tfrac{1}{2}\Pi\right) + \delta^\alpha \Pi_{\mu\alpha} = 0. \tag{49}$$

4.2. Gravito-electromagnetism

The 1+1+2 GEM system is of prime importance as this paper is predominately focused on decoupling the GEM 2-tensor harmonic amplitudes. The once contracted Bianchi identities may be written in terms of the Weyl and energy–momentum tensor according to

$$B_{\nu\sigma\tau} := \nabla^\mu C_{\mu\nu\sigma\tau} - \left[\nabla_{[\sigma} T_{\tau]\nu} + \tfrac{1}{3} g_{\nu[\sigma}\nabla_{\tau]} T\right] = 0. \tag{50}$$

Before proceeding with the linearized system, we momentarily discuss the fully non-linear 1+3 GEM system, for which it is important to note that it is invariant under the simultaneous transformation $E_{\mu\nu} \to H_{\mu\nu}$ and $H_{\mu\nu} \to -E_{\mu\nu}$ (in the absence of sources).

In a recent paper [7], we used linear algebra techniques to show that the most natural way to decouple a system with these particular invariance properties is to choose new complex dynamical variables. This has also been discussed elsewhere; for example, see [10] where they introduce a complex tensor defined as $\mathcal{I}_{\mu\nu} := E_{\mu\nu} \pm \mathrm{i}H_{\mu\nu}$ (where i is the complex number). It was also this reason why we successfully decoupled the EM 2-vector harmonic amplitudes in [8].

We now turn the attention to the first-order 1+1+2 GEM system which reduces to[2]

$$\delta\left[\left(\mathcal{L}_n + \tfrac{3}{2}\phi\right)\mathcal{E}\right] + \delta^\alpha \mathcal{E}_\alpha = \Re[\mathcal{G}], \tag{51}$$

$$\left(\mathcal{L}_n + \tfrac{3}{2}\phi\right)\mathcal{H} + \delta^\alpha \mathcal{H}_\alpha + 3\mathcal{E}\Omega = \Im[\mathcal{G}], \tag{52}$$

$$\delta\left[\left(\mathcal{L}_u - \tfrac{3}{2}\Sigma + \theta\right)\mathcal{E}\right] - \epsilon^{\alpha\beta}\delta_\alpha \mathcal{H}_\beta = \Re[\mathcal{F}], \tag{53}$$

[1] These are derived as follows, (47) from $u^\alpha \nabla^\beta T_{\alpha\beta} = 0$; (48) from $n^\alpha \nabla^\beta T_{\alpha\beta} = 0$ and (49) from $\nabla^\alpha T_{\bar{\mu}\alpha} = 0$.

[2] These are derived as follows: (51) from $u^\alpha u^\beta n^\gamma B_{\alpha\beta\gamma} = 0$; (52) from $\epsilon^{\beta\gamma} u^\alpha B_{\alpha\beta\gamma} = 0$; (53) from $u^\alpha n^\beta n^\gamma B_{\beta\gamma\alpha} = 0$; (54) from $\epsilon^{\alpha\beta} n^\gamma B_{\gamma\alpha\beta} = 0$; (55) from $u^\beta u^\gamma B_{\bar{\mu}\beta\gamma} = 0$; (56) from $\epsilon_{\bar{\mu}}{}^{\beta\gamma} u^\alpha B_{\alpha\beta\gamma} = 0$; (57) from $n^\nu u^\gamma B_{(\bar{\mu}\nu)\gamma} = 0$; (58) from $n^\nu \epsilon_{(\bar{\mu}}{}^{\alpha\beta} B_{\nu)\alpha\beta} = 0$; (59) from $u^\alpha B_{(\bar{\mu}\bar{\nu})\alpha} = 0$ and (60) from $\epsilon_{(\bar{\mu}}{}^{\alpha\beta} B_{\bar{\nu})\alpha\beta} = 0$.

$$\left(\mathcal{L}_u - \tfrac{3}{2}\Sigma + \theta\right)\mathcal{H} + \epsilon^{\alpha\beta}\delta_\alpha\mathcal{E}_\beta + 3\mathcal{E}\xi = \Im[\mathcal{F}], \tag{54}$$

$$(\mathcal{L}_n + \phi)\mathcal{E}_{\bar\mu} + \delta^\alpha\mathcal{E}_{\mu\alpha} - \tfrac{1}{2}X_\mu + \tfrac{3}{2}\Sigma\epsilon_\mu{}^\alpha\mathcal{H}_\alpha + \tfrac{3}{2}\mathcal{E}a_\mu = \Re[\mathcal{G}_\mu], \tag{55}$$

$$(\mathcal{L}_n + \phi)\mathcal{H}_{\bar\mu} + \delta^\alpha\mathcal{H}_{\mu\alpha} - \tfrac{1}{2}\delta_\mu\mathcal{H} - \tfrac{3}{2}\Sigma\epsilon_\mu{}^\alpha\mathcal{E}_\alpha + \tfrac{3}{2}\mathcal{E}\epsilon_\mu{}^\alpha\left(\Sigma_\alpha + \epsilon_\alpha{}^\beta\Omega_\beta\right) = \Im[\mathcal{G}_\mu], \tag{56}$$

$$\left(\mathcal{L}_u - \Sigma + \tfrac{2}{3}\theta\right)\mathcal{E}_{\bar\mu} - \epsilon_\mu{}^\alpha\delta^\beta\mathcal{H}_{\alpha\beta} - \tfrac{1}{2}\epsilon_\mu{}^\alpha\left[\delta_\alpha\mathcal{H} - (2\mathcal{A} - \phi)\mathcal{H}_\alpha\right] + \tfrac{3}{2}\mathcal{E}\alpha_\mu = \Re[\mathcal{F}_\mu], \tag{57}$$

$$\left(\mathcal{L}_u - \Sigma + \tfrac{2}{3}\theta\right)\mathcal{H}_{\bar\mu} + \epsilon_\mu{}^\alpha\delta^\beta\mathcal{E}_{\alpha\beta} + \tfrac{1}{2}\epsilon_\mu{}^\alpha\left[X_\alpha - (2\mathcal{A} - \phi)\mathcal{E}_\alpha\right] + \tfrac{3}{2}\mathcal{E}\epsilon_\mu{}^\alpha\mathcal{A}_\alpha = \Im[\mathcal{F}_\mu], \tag{58}$$

$$\left(\mathcal{L}_u + \tfrac{5}{2}\Sigma + \tfrac{1}{3}\theta\right)\mathcal{E}_{\bar\mu\bar\nu} + \epsilon_{(\mu}{}^\alpha\left(\mathcal{L}_n + 2\mathcal{A} - \tfrac{1}{2}\phi\right)\mathcal{H}_{\nu)\alpha} - \epsilon_{\{\mu}{}^\alpha\delta_{|\alpha|}\mathcal{H}_{\nu\}} + \tfrac{3}{2}\mathcal{E}\Sigma_{\mu\nu} = \Re[\mathcal{F}_{\mu\nu}], \tag{59}$$

$$\left(\mathcal{L}_u + \tfrac{5}{2}\Sigma + \tfrac{1}{3}\theta\right)\mathcal{H}_{\bar\mu\bar\nu} - \epsilon_{(\mu}{}^\alpha\left(\mathcal{L}_n + 2\mathcal{A} - \tfrac{1}{2}\phi\right)\mathcal{E}_{\nu)\alpha} + \epsilon_{\{\mu}{}^\alpha\delta_{|\alpha|}\mathcal{E}_{\nu\}} + \tfrac{3}{2}\mathcal{E}\epsilon_{(\mu}{}^\alpha\zeta_{\nu)\alpha} = \Im[\mathcal{F}_{\mu\nu}]. \tag{60}$$

The first-order energy–momentum source terms have been suitably defined in a complex form for later convenience as

$$\mathcal{F} := -\tfrac{1}{2}(\mu + p)\Sigma - \tfrac{1}{3}\left(\mathcal{L}_n + 2\mathcal{A} - \tfrac{1}{2}\phi\right)\mathcal{Q} + \tfrac{1}{6}\delta^\alpha\mathcal{Q}_\alpha - \tfrac{1}{2}\left(\mathcal{L}_u + \tfrac{1}{2}\Sigma + \tfrac{1}{3}\theta\right)\Pi + \mathrm{i}\tfrac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha\Pi_\beta, \tag{61}$$

$$\mathcal{G} := \tfrac{1}{3}\mathcal{L}_n\mu + \tfrac{1}{2}\mathcal{Q}\left(\Sigma - \tfrac{2}{3}\theta\right) - \tfrac{1}{2}\delta^\alpha\Pi_\alpha - \tfrac{1}{2}\left(\mathcal{L}_n + \tfrac{3}{2}\phi\right)\Pi - \mathrm{i}\tfrac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha\mathcal{Q}_\beta, \tag{62}$$

$$\mathcal{F}_\mu := -\tfrac{1}{2}\left[\mathcal{L}_u\Pi_{\bar\mu} + \left(\mathcal{A} - \tfrac{1}{2}\phi\right)\mathcal{Q}_{\bar\mu} + \delta_\mu\mathcal{Q}\right] + \mathrm{i}\tfrac{1}{2}\epsilon_\mu{}^\alpha\left[\tfrac{1}{3}\delta_\alpha(\mu + 3\Pi) - \left(\Sigma + \tfrac{1}{3}\theta\right)\mathcal{Q}_\alpha - \left(\mathcal{L}_n + \tfrac{1}{2}\phi\right)\Pi_\alpha\right], \tag{63}$$

$$\mathcal{G}_\mu := \tfrac{1}{3}\delta_\mu\left(\mu + \tfrac{3}{4}\Pi\right) - \tfrac{1}{4}\left(\Sigma + \tfrac{4}{3}\theta\right)\mathcal{Q}_\mu - \tfrac{1}{2}(\mathcal{L}_n + \phi)\Pi_{\bar\mu} - \tfrac{1}{2}\delta^\alpha\Pi_{\mu\alpha} + \mathrm{i}\tfrac{1}{2}\epsilon_\mu{}^\alpha\left(\mathcal{L}_n\mathcal{Q}_\alpha - \delta_\alpha\mathcal{Q} + \tfrac{3}{2}\Sigma\Pi_\alpha\right), \tag{64}$$

$$\mathcal{F}_{\mu\nu} := -\tfrac{1}{2}\delta_{\{\mu}\mathcal{Q}_{\nu\}} - \tfrac{1}{2}\left(\mathcal{L}_u + \tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta\right)\Pi_{\bar\mu\bar\nu} + \mathrm{i}\tfrac{1}{2}\left[\epsilon_{\{\mu}{}^\alpha\delta_{|\alpha|}\Pi_{\nu\}} - \epsilon_{(\mu}{}^\alpha\left(\mathcal{L}_n - \tfrac{1}{2}\phi\right)\Pi_{\bar\nu)\alpha}\right]. \tag{65}$$

The first-order GEM system (51)–(60) generalizes those given in [1] in two significant ways; they generalize from the Schwarzschild perturbations towards an arbitrary vacuum LRS class II spacetime and they also generalize from the vacuum energy–momentum perturbations towards a full energy–momentum perturbation. Furthermore, a very recent independent study of these equations for LRS spacetimes has been carried out in [23]. We have also taken a lot of care to ensure that all quantities are gauge-invariant; for example, the first-order term in (51), $\delta\left[\left(\mathcal{L}_n + \frac{3}{2}\phi\right)\mathcal{E}\right]$, is gauge-invariant as its corresponding background term vanishes according to (16), i.e. $\left(\mathcal{L}_n + \frac{3}{2}\phi\right)\mathcal{E} = 0$. However, we now choose to rewrite (51)–(54) in terms of the 2-gradient quantity X_μ defined in (26). Thus, new complex variables are chosen according to the invariance properties of the 1+3 GEM system discussed above and, without loss of generality, we write the GEM system in a new 1+1+2 complex form,

$$\left(\mathcal{L}_n + \tfrac{3}{2}\phi\right)\mathcal{C}_\mu + \delta_\mu\delta^\alpha\Phi_\alpha + \tfrac{3}{2}\mathcal{E}\left[Y_\mu - \phi a_\mu - 2\left(\Sigma - \tfrac{2}{3}\theta\right)\epsilon_\mu{}^\alpha\Omega_\alpha + \mathrm{i}\,2\delta_\mu\Omega\right] = \delta_\mu\mathcal{G}, \tag{66}$$

$$\left(\mathcal{L}_u - \tfrac{3}{2}\Sigma + \theta\right)\mathcal{C}_{\bar\mu} + \mathrm{i}\,\delta_\mu(\epsilon^{\alpha\beta}\delta_\alpha\Phi_\beta) - \tfrac{3}{2}\mathcal{E}\left[\mathcal{A}_\mu\left(\Sigma - \tfrac{2}{3}\theta\right) + \phi(\Sigma_\mu - \epsilon_\mu{}^\alpha\Omega_\alpha + \alpha_\mu) + W_\mu - \mathrm{i}\,2\delta_\mu\xi\right] = \delta_\mu\mathcal{F}, \tag{67}$$

$$(\mathcal{L}_n + \phi)\Phi_{\bar\mu} + \delta^\alpha\Phi_{\mu\alpha} - \tfrac{1}{2}\delta(\delta_\mu\Phi) - \mathrm{i}\,\tfrac{3}{2}\Sigma\epsilon_\mu{}^\alpha\Phi_\alpha + \tfrac{3}{2}\mathcal{E}\Lambda_\mu = \mathcal{G}_\mu, \tag{68}$$

$$\left(\mathcal{L}_u - \Sigma + \tfrac{2}{3}\theta\right)\Phi_{\bar\mu} + \mathrm{i}\epsilon_\mu{}^\alpha\delta^\beta\Phi_{\alpha\beta} + \mathrm{i}\,\tfrac{1}{2}\epsilon_\mu{}^\alpha\left[\mathcal{C}_\alpha - (2\mathcal{A} - \phi)\Phi_\alpha\right] + \tfrac{3}{2}\mathcal{E}\Upsilon_\mu = \mathcal{F}_\mu, \tag{69}$$

$$\left(\mathcal{L}_u + \tfrac{5}{2}\Sigma + \tfrac{1}{3}\theta\right)\Phi_{\bar{\mu}\bar{\nu}} - \mathrm{i}\epsilon_{(\mu}{}^{\alpha}\left(\mathcal{L}_n + 2\mathcal{A} - \tfrac{1}{2}\phi\right)\Phi_{\nu)\alpha} + \mathrm{i}\epsilon_{\{\mu}{}^{\alpha}\delta_{|\alpha|}\Phi_{\nu\}} + \tfrac{3}{2}\mathcal{E}\Lambda_{\mu\nu} = \mathcal{F}_{\mu\nu}, \tag{70}$$

where

$$\mathcal{C}_\mu := X_\mu + \mathrm{i}\,\delta_\mu\mathcal{H}, \qquad \Phi_\mu := \mathcal{E}_\mu + \mathrm{i}\,\mathcal{H}_\mu \qquad \text{and} \qquad \Phi_{\mu\nu} := \mathcal{E}_{\mu\nu} + \mathrm{i}\,\mathcal{H}_{\mu\nu}. \tag{71}$$

[3] Furthermore, whilst constructing these complex equations, several other terms naturally combine and therefore, three new complex definitions are

$$\Upsilon_\mu := \alpha_\mu + \mathrm{i}\epsilon_\mu{}^{\alpha}\mathcal{A}_\alpha, \qquad \Lambda_\mu := a_\mu + \mathrm{i}\epsilon_\mu{}^{\alpha}\left(\Sigma_\alpha + \epsilon_\alpha{}^{\beta}\Omega_\beta\right) \quad \text{and} \quad \Lambda_{\mu\nu} := \Sigma_{\mu\nu} + \mathrm{i}\epsilon_{(\mu}{}^{\alpha}\zeta_{\nu)\alpha}. \tag{72}$$

In section 5 we will use the complex GEM system (66)–(70) to fully decouple the complex GEM 2-tensor, $\Phi_{\mu\nu}$, from all the remaining 1+1+2 quantities.

4.3. The 1+1+2 Ricci identities

The Ricci identities for both u^μ and n^μ are defined conveniently as

$$Q_{\mu\nu\sigma} := 2\nabla_{[\mu}\nabla_{\nu]}u_\sigma - R_{\mu\nu\sigma\tau}u^\tau = 0, \tag{73}$$

$$R_{\mu\nu\sigma} := 2\nabla_{[\mu}\nabla_{\nu]}n_\sigma - R_{\mu\nu\sigma\tau}n^\tau = 0, \tag{74}$$

where $R_{\mu\nu\sigma\tau}$ is the Riemann tensor. We now linearize these, reduce them to 1+1+2 form and categorize them into constraint, propagation, transportation and evolution equations. We also make two new definitions for combinations that arise quite frequently,

$$\lambda_\mu := \Sigma_\mu - \epsilon_\mu{}^{\alpha}\Omega_\mu \qquad \text{and} \qquad \upsilon_\mu := \Sigma_\mu + \epsilon_\mu{}^{\alpha}\Omega_\mu, \tag{75}$$

such that the following system can be written in a more readable form.

- Constraint equations[4]

$$W_\mu + \phi\lambda_\mu + 2\delta^\alpha\Sigma_{\mu\alpha} + 2\epsilon_\mu{}^{\alpha}\mathcal{H}_\alpha + 2\epsilon_\mu{}^{\alpha}\delta_\alpha\Omega = -\mathcal{Q}_\mu, \tag{76}$$

$$Y_\mu - 2\epsilon_\mu{}^{\alpha}\delta_\alpha\xi - 2\delta^\alpha\zeta_{\mu\alpha} + 2\mathcal{E}_\mu + \left(\Sigma - \tfrac{2}{3}\theta\right)\lambda_\mu = -\Pi_\mu, \tag{77}$$

$$\epsilon^{\alpha\beta}\delta_\alpha\lambda_\beta - (2\mathcal{A} - \phi)\Omega + 3\xi\Sigma - \mathcal{H} = 0. \tag{78}$$

- Propagation equations[5]

$$\delta\left\{\left(\mathcal{L}_n + \tfrac{1}{2}\phi\right)\phi + \left(\Sigma + \tfrac{1}{3}\theta\right)\left(\Sigma - \tfrac{2}{3}\theta\right) + \mathcal{E}\right\} - \delta^\alpha a_\alpha = -\tfrac{2}{3}\mu - \tfrac{1}{2}\Pi, \tag{79}$$

$$\delta\left\{\mathcal{L}_n\left(\Sigma - \tfrac{2}{3}\theta\right) + \tfrac{3}{2}\phi\Sigma\right\} + \delta^\alpha\upsilon_\alpha = -\mathcal{Q}, \tag{80}$$

$$(\mathcal{L}_n + \phi)\xi - \left(\Sigma + \tfrac{1}{3}\theta\right)\Omega - \tfrac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha a_\beta = 0, \tag{81}$$

$$(\mathcal{L}_n - \mathcal{A} + \phi)\Omega + \delta^\alpha\Omega_\alpha = 0, \tag{82}$$

$$\mathcal{L}_n\lambda_{\bar{\mu}} + \tfrac{1}{2}\phi\upsilon_\mu - 2\mathcal{A}\epsilon_\mu{}^{\alpha}\Omega_\alpha - \delta_\mu\left(\Sigma + \tfrac{1}{3}\theta\right) + \tfrac{3}{2}\Sigma a_\mu - \epsilon_\mu{}^{\alpha}\mathcal{H}_\alpha = -\tfrac{1}{2}\mathcal{Q}_\mu, \tag{83}$$

$$\left(\mathcal{L}_n - \tfrac{1}{2}\phi\right)\Sigma_{\bar{\mu}\bar{\nu}} - \tfrac{3}{2}\Sigma\zeta_{\mu\nu} - \epsilon_{(\mu}{}^{\alpha}\mathcal{H}_{\nu)\alpha} - \delta_{\{\mu}\upsilon_{\nu\}} = 0, \tag{84}$$

$$\mathcal{L}_n\zeta_{\bar{\mu}\bar{\nu}} - \left(\Sigma + \tfrac{1}{3}\theta\right)\Sigma_{\mu\nu} + \mathcal{E}_{\mu\nu} - \delta_{\{\mu}a_{\nu\}} = -\tfrac{1}{2}\Pi_{\mu\nu}. \tag{85}$$

[3] It is also possible to choose the complex conjugates, i.e. $\Phi^*_{\mu\nu}$, Φ^*_μ and Φ^* and the corresponding governing equations are simply found by taking the complex conjugate of the equations governing $\Phi_{\mu\nu}$, Φ_μ and Φ.

[4] (76) from a combination of $n^\mu u^\sigma R_{\mu\bar{\nu}\sigma} = 0$, $N^{\mu\sigma}Q_{\mu\bar{\nu}\sigma} = 0$ and $n^\mu n^\sigma Q_{\mu\bar{\nu}\sigma} = 0$; (77) from $N^{\nu\sigma}R_{\bar{\mu}\nu\sigma} = 0$ and (78) from $\epsilon^{\mu\nu}u^\sigma R_{\mu\nu\sigma} = 0$.

[5] (79) from $n^\mu N^{\nu\sigma}R_{\mu\nu\sigma} = 0$; (80) from $n^\mu N^{\nu\sigma}Q_{\mu\nu\sigma} = 0$; (81) from $n^\mu\epsilon^{\nu\sigma}R_{\mu\nu\sigma} = 0$; (82) from $\epsilon^{\mu\nu\sigma}Q_{\mu\nu\sigma} = 0$; (83) from $D^\alpha\sigma_{\mu\alpha}$ equation and $n^\mu u^\sigma R_{\mu\bar{\nu}\sigma} = 0$; (84) from $n^\mu Q_{\mu(\bar{\nu}\bar{\sigma})} = 0$ and (85) from $n^\mu R_{\mu(\bar{\nu}\bar{\sigma})} = 0$.

- Transportation[6]

$$\delta\{(\mathcal{L}_u + \Sigma + \tfrac{1}{3}\theta)(\Sigma + \tfrac{1}{3}\theta) - (\mathcal{L}_n + \mathcal{A})\mathcal{A} + \mathcal{E}\} = -\tfrac{1}{6}(\mu + 3p - 3\Pi), \quad (86)$$

$$(\mathcal{L}_u + \Sigma + \tfrac{1}{3}\theta)\upsilon_{\bar{\mu}} - (\mathcal{L}_n + \mathcal{A} - \tfrac{1}{2}\phi)\mathcal{A}_{\bar{\mu}} - \mathcal{A}a_\mu + \tfrac{3}{2}\Sigma\alpha_\mu + \mathcal{E}_\mu = \tfrac{1}{2}\Pi_\mu, \quad (87)$$

$$(\mathcal{L}_u + \tfrac{3}{2}\Sigma)a_{\bar{\mu}} - (\mathcal{L}_n + \mathcal{A})\alpha_{\bar{\mu}} - (\mathcal{A} - \tfrac{1}{2}\phi)\upsilon_\mu + (\Sigma + \tfrac{1}{3}\theta)\mathcal{A}_\mu - \epsilon_\mu{}^\alpha\mathcal{H}_\alpha = -\tfrac{1}{2}\mathcal{Q}_\mu. \quad (88)$$

- Evolution equations[7]

$$\delta\{(\mathcal{L}_u - \tfrac{1}{2}\Sigma + \tfrac{1}{3}\theta)\phi + \mathcal{A}(\Sigma - \tfrac{2}{3}\theta)\} - \delta^\gamma\alpha_\gamma = \mathcal{Q}, \quad (89)$$

$$\delta\{(\mathcal{L}_u - \tfrac{1}{2}\Sigma + \tfrac{1}{3}\theta)(\Sigma - \tfrac{2}{3}\theta) + \mathcal{A}\phi + \mathcal{E}\} + \delta^\alpha\mathcal{A}_\alpha = \tfrac{1}{3}(\mu + 3p + \tfrac{3}{2}\Pi), \quad (90)$$

$$(\mathcal{L}_u - \tfrac{1}{2}\Sigma + \tfrac{1}{3}\theta)\xi - \tfrac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha\alpha_\beta - (\mathcal{A} - \tfrac{1}{2}\phi)\Omega - \tfrac{1}{2}\mathcal{H} = 0, \quad (91)$$

$$(\mathcal{L}_u - \Sigma - \tfrac{2}{3}\theta)\Omega - \mathcal{A}\xi - \tfrac{1}{2}\epsilon^{\alpha\beta}\delta_\alpha\mathcal{A}_\beta = 0, \quad (92)$$

$$(\mathcal{L}_u + \theta)\lambda_{\bar{\mu}} - Z_\mu - (\mathcal{A} - \tfrac{1}{2}\phi)\mathcal{A}_\mu + \tfrac{3}{2}\Sigma\alpha_\mu + \mathcal{E}_\mu = \tfrac{1}{2}\Pi_\mu, \quad (93)$$

$$(\mathcal{L}_u + \tfrac{1}{2}\Sigma - \tfrac{1}{3}\theta)\zeta_{\bar{\mu}\bar{\nu}} - (\mathcal{A} - \tfrac{1}{2}\phi)\Sigma_{\mu\nu} - \epsilon_{(\mu}{}^\alpha\mathcal{H}_{\nu)\alpha} - \delta_{\{\mu}\alpha_{\nu\}} = 0, \quad (94)$$

$$\mathcal{L}_u\Sigma_{\bar{\mu}\bar{\nu}} - \mathcal{A}\zeta_{\mu\nu} - \delta_{\{\mu}\mathcal{A}_{\nu\}} + \mathcal{E}_{\mu\nu} = \tfrac{1}{2}\Pi_{\mu\nu}. \quad (95)$$

Similarly, these 1+1+2 Ricci identities (76)–(95) are again a significant generalization of the results in [1]. They now include full energy–momentum sources and moreover, they are for arbitrary vacuum LRS class II spacetimes. Moreover, the very recent independent study by Clarkson [23] presents the equations for LRS spacetimes. For the subsequent decoupling of the complex GEM 2-tensor, we require evolution, transportation and propagation equations for the complex variables defined in (72)[8]

$$\begin{aligned}(\mathcal{L}_u + \tfrac{3}{2}\Sigma)\Lambda_{\bar{\mu}} - (\mathcal{L}_n + \mathcal{A})\Upsilon_{\bar{\mu}} + \mathrm{i}\epsilon_\mu{}^\alpha\Phi_\alpha - \mathrm{i}\,\mathcal{A}\epsilon_\mu{}^\alpha\Lambda_\alpha + \tfrac{1}{2}\phi(\upsilon_\mu + \mathrm{i}\epsilon_\mu{}^\alpha\mathcal{A}_\alpha)\\ + (\Sigma + \tfrac{1}{3}\theta)\mathcal{A}_\mu - \mathrm{i}\tfrac{1}{2}(\Sigma - \tfrac{2}{3}\theta)\epsilon_\mu{}^\alpha\upsilon_\alpha + \mathrm{i}\tfrac{3}{2}\Sigma\epsilon_\mu{}^\alpha\alpha_\alpha = -\tfrac{1}{2}(\mathcal{Q}_\mu - \mathrm{i}\epsilon_\mu{}^\alpha\Pi_\alpha),\end{aligned} \quad (96)$$

$$\begin{aligned}\mathcal{L}_u\Lambda_{\bar{\mu}\bar{\nu}} + \Phi_{\mu\nu} - \mathrm{i}\,\mathcal{A}\epsilon_{(\mu}{}^\alpha\Lambda_{\nu)\alpha} + \mathrm{i}\tfrac{1}{2}\phi\epsilon_{(\mu}{}^\alpha\Sigma_{\nu)\alpha}\\ + \mathrm{i}\tfrac{1}{2}(\Sigma - \tfrac{2}{3}\theta)\epsilon_{(\mu}{}^\alpha\zeta_{\nu)\alpha} - \mathrm{i}\epsilon_{\{\mu}{}^\alpha\delta_{\nu\}}\Upsilon_\alpha = \tfrac{1}{2}\Pi_{\mu\nu},\end{aligned} \quad (97)$$

$$\begin{aligned}\mathcal{L}_n\Lambda_{\bar{\mu}\bar{\nu}} + \mathrm{i}\epsilon_{(\mu}{}^\alpha\Phi_{\nu)\alpha} - \mathrm{i}(\Sigma + \tfrac{1}{3}\theta)\epsilon_{(\mu}{}^\alpha\Sigma_{\nu)\alpha} - \tfrac{3}{2}\Sigma\zeta_{\mu\nu}\\ - \tfrac{1}{2}\phi\Sigma_{\mu\nu} - \mathrm{i}\epsilon_{\{\mu}{}^\alpha\delta_{\nu\}}\Lambda_\alpha = -\mathrm{i}\tfrac{1}{2}\epsilon_{(\mu}{}^\alpha\Pi_{\nu)\alpha}.\end{aligned} \quad (98)$$

4.4. Commutation relationships

Finally, we present how the various derivatives defined in this paper commute and generalize the results from [2],

$$(\mathcal{L}_u + \Sigma + \tfrac{1}{3}\theta)\mathcal{L}_n\Phi_{\bar{\mu}\ldots\bar{\nu}} - (\mathcal{L}_n + \mathcal{A})\mathcal{L}_u\Phi_{\bar{\mu}\ldots\bar{\nu}} = 0, \quad (99)$$

$$\mathcal{L}_u\delta_\sigma\Phi_{\bar{\mu}\ldots\bar{\nu}} - \delta_\sigma\mathcal{L}_u\Phi_{\bar{\mu}\ldots\bar{\nu}} = 0, \quad (100)$$

$$\mathcal{L}_n\delta_\sigma\Phi_{\bar{\mu}\ldots\bar{\nu}} - \delta_\sigma\mathcal{L}_n\Phi_{\bar{\mu}\ldots\bar{\nu}} = 0, \quad (101)$$

[6] (86) from $u^\mu n^\nu u^\sigma R_{\mu\nu\sigma}$; (87) from $n^\mu u^\nu N_\sigma{}^\gamma Q_{\mu\nu\gamma} = 0$ and (88) from $u^\alpha n^\beta R_{\alpha\beta\bar{\mu}} = 0$.

[7] (89) from $u^\mu N^{\nu\sigma} R_{\mu\nu\sigma} = 0$; (90) from $u^\mu N^{\nu\sigma} Q_{\mu\nu\sigma} = 0$; (91) from $u^\mu\epsilon^{\nu\sigma} R_{\mu\nu\sigma} = 0$; (92) from $u^\mu\epsilon^{\nu\sigma} Q_{\mu\nu\sigma} = 0$; (93) from $u^\mu n^\sigma N_\nu{}^\alpha Q_{\mu\alpha\sigma} = 0$;(94) from $u^\mu R_{\mu(\bar{\nu}\bar{\sigma})} = 0$; (95) from $u^\mu Q_{\mu(\bar{\nu}\bar{\sigma})} = 0$.

[8] (96) from (87) and (88); (97) from (94), (95); (98) from (84) and (85).

where $\Phi_{\mu\ldots\nu}$ represents a first-order scalar, first-order 2-vector and a first-order 2-tensor. The commutators not only play a vital role in decoupling the equations at hand, they also provide a rigorous test that the equations present here are correct and accurate. Every equation (61)–(70) and (76)–(98) has been checked to satisfy all of the commutator relationships (99)–(100) and this is inclusive of careful checks of all energy–momentum source terms (61)–(65).

5. Decoupling the complex GEM 2-tensor and its tensor harmonic amplitudes

We use the complex 1+1+2 Bianchi identities (66)–(70) to construct a new, covariant and gauge-invariant equation governing the first-order complex GEM 2-tensor $\Phi_{\mu\nu}$. This is with a complete description of the covariant and gauge-invariant, first-order energy–momentum sources. It begins by taking the Lie derivative with respect to u^μ of (70). It is then required to use the commutation relationships (99)–(100) followed by substitutions of (68) through to (70). Finally, (97) and (98) are used for further simplifications to obtain

$$\begin{aligned}&[(\mathcal{L}_u+\theta)\mathcal{L}_u-(\mathcal{L}_n+\mathcal{A}+\phi)\mathcal{L}_n-V]\Phi_{\mu\nu}\\&\qquad-\mathrm{i}\epsilon_{(\mu}{}^{\alpha}\left[(4\mathcal{A}-2\phi)\mathcal{L}_u-6\Sigma\mathcal{L}_n+U\right]\Phi_{\nu)\alpha}=\mathcal{M}_{\mu\nu}.\end{aligned}\tag{102}$$

The two background scalars related to the potential, and the first-order energy–momentum source, have been defined respectively,

$$V:=\delta^2+8\mathcal{E}-4\mathcal{A}^2+4\mathcal{A}\phi-\phi^2+9\Sigma^2-3\Lambda,\tag{103}$$

$$U:=2\big(\mathcal{L}_u-\Sigma+\tfrac{2}{3}\theta\big)\mathcal{A}-3\big(\mathcal{L}_n+\tfrac{7}{6}\phi\big)\Sigma-\tfrac{2}{3}\theta\phi-2\Lambda,\tag{104}$$

$$\mathcal{M}_{\mu\nu}:=\big(\mathcal{L}_u-\tfrac{5}{2}\Sigma+\tfrac{2}{3}\theta\big)\mathcal{F}_{\bar\mu\bar\nu}+\mathrm{i}\epsilon_{(\mu}{}^{\alpha}\big(\mathcal{L}_n-\mathcal{A}+\tfrac{3}{2}\phi\big)\mathcal{F}_{\nu)\alpha}-\mathrm{i}\epsilon_{\{\mu}{}^{\alpha}\delta_{|\alpha|}\mathcal{F}_{\nu\}}-\delta_{\{\mu}\mathcal{G}_{\nu\}}.\tag{105}$$

It was possible to eliminate all Lie derivatives in V and write it explicitly as algebraic combinations of the background LRS class II scalars. However, the Lie derivatives in the other potential term, U, cannot be reduced any further because there is no evolution equation for $\mathcal{A}$.

Thus (102) demonstrates that, for arbitrary vacuum LRS class II spacetimes, the complex GEM 2-tensor decouples from the remaining GEM and 1+1+2 quantities. We next show how this 2-tensor decouples further by using a tensor harmonic expansion, but we first take a closer inspection of the energy–momentum source, $\mathcal{M}_{\mu\nu}$,

$$\begin{aligned}\mathcal{M}_{\mu\nu}=-\tfrac{1}{2}\big\{&\big(\mathcal{L}_u-2\Sigma+\tfrac{1}{3}\theta\big)\mathcal{L}_u\Pi_{\mu\nu}+(\mathcal{L}_n-\mathcal{A}+\phi)\mathcal{L}_n\Pi_{\mu\nu}-\mathcal{M}\Pi_{\mu\nu}-2\delta_{\{\mu}\delta^{\alpha}\Pi_{\nu\}\alpha}\big\}\\&+2(\mathcal{L}_n+\phi)\delta_{\{\mu}\Pi_{\nu\}}+2\big(\Sigma+\tfrac{1}{3}\theta\big)\delta_{\{\mu}\mathcal{Q}_{\nu\}}+\tfrac{1}{2}\delta_{\{\mu}\delta_{\nu\}}(p+2\Pi),\\&+\mathrm{i}\epsilon_{\{\mu}{}^{\alpha}\big\{-\big(\mathcal{L}_u-\tfrac{1}{2}\Sigma+\tfrac{1}{3}\theta\big)\mathcal{L}_n\Pi_{\nu\}\alpha}+\big(\mathcal{A}-\tfrac{1}{2}\phi\big)\mathcal{L}_u\Pi_{\nu\}\alpha}+\phi\big(\Sigma-\tfrac{2}{3}\theta\big)\Pi_{\nu\}\alpha}\\&+\big(\mathcal{L}_u-2\Sigma+\tfrac{1}{3}\theta\big)\delta_{\nu\}}\Pi_\alpha-(\mathcal{L}_n-\mathcal{A}+\phi)\delta_{\nu\}}\mathcal{Q}_\alpha+\delta_{\nu\}}\delta_\alpha\mathcal{Q}\big\},\end{aligned}\tag{106}$$

where

$$\mathcal{M}=\tfrac{1}{2}\big(\Sigma-\tfrac{2}{3}\theta\big)^2+\tfrac{1}{2}\mathcal{A}\phi+\tfrac{1}{2}\phi^2-\tfrac{1}{2}\mathcal{E}.\tag{107}$$

It is interesting to see which energy–momentum terms play an important role in the evolution and propagation of the complex GEM 2-tensor. By considering the ‘principle part’, or the parts which involve second-order Lie derivatives, it seems that the first-order anisotropic stress may have a predominate influence here.

5.1. Decoupling the complex GEM 2-tensor harmonic amplitudes

The complex GEM tensor, $\Phi_{\mu\nu}$, and the energy–momentum source, $\mathcal{M}_{\mu\nu}$, are expanded using tensor harmonics according to

$$\Phi_{\mu\nu} = \Phi_{\rm T} Q_{\mu\nu} + \bar{\Phi}_{\rm T} \bar{Q}_{\mu\nu} \qquad \text{and} \qquad \mathcal{M}_{\mu\nu} = \mathcal{M}_{\rm T} Q_{\mu\nu} + \bar{\mathcal{M}}_{\rm T} \bar{Q}_{\mu\nu}.$$

Consequently, (102) results in two coupled equations of the form

$$\begin{aligned}&\left[\left(\mathcal{L}_u - 2\Sigma + \tfrac{7}{3}\theta\right)\mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 3\phi)\mathcal{L}_n - \tilde{V}\right]\Phi_{\rm T}\\ &\qquad + {\rm i}\,[6\Sigma\mathcal{L}_n - (4\mathcal{A} - 2\phi)\mathcal{L}_u - \tilde{U}]\bar{\Phi}_{\rm T} = \mathcal{M}_{\rm T},\end{aligned} \tag{108}$$

$$\begin{aligned}&\left[\left(\mathcal{L}_u - 2\Sigma + \tfrac{7}{3}\theta\right)\mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 3\phi)\mathcal{L}_n - \tilde{V}\right]\bar{\Phi}_{\rm T}\\ &\qquad - {\rm i}\,[6\Sigma\mathcal{L}_n - (4\mathcal{A} - 2\phi)\mathcal{L}_u - \tilde{U}]\Phi_{\rm T} = \bar{\mathcal{M}}_{\rm T},\end{aligned} \tag{109}$$

where new potential terms are defined,

$$\begin{aligned}\tilde{V} &:= -\frac{k^2}{r^2} + 2\mathcal{E} - 4\mathcal{A}^2 + 4\mathcal{A}\phi + \frac{3}{2}\phi^2 + \frac{13}{2}\Sigma^2 - \frac{10}{9}\theta^2 + \frac{10}{3}\Sigma\theta,\\ \tilde{U} &:= 2(\mathcal{L}_u - 3\Sigma + 2\theta)\mathcal{A} - 3\left(\mathcal{L}_n + \frac{5}{2}\phi\right)\Sigma - 2\theta\phi.\end{aligned} \tag{110}$$

By inspecting the coupled system (108) and (109), it is clear that they are invariant under the simultaneous transformation of $\Phi_{\rm T} \to \bar{\Phi}_{\rm T}$ and $\bar{\Phi}_{\rm T} \to -\Phi_{\rm T}$, and similarly for the sources, $\mathcal{M}_{\rm T} \to \bar{\mathcal{M}}_{\rm T}$ and $\bar{\mathcal{M}}_{\rm T} \to -\mathcal{M}_{\rm T}$. Thus, the coupled system (108)–(109) is precisely of the form as discussed at the beginning of section 4.2. Therefore, they will decouple quite naturally by constructing two new complex dependent variables,

$$\Phi_+ := \Phi_{\rm T} + {\rm i}\,\bar{\Phi}_{\rm T} \qquad \text{and} \qquad \Phi_- := \Phi_{\rm T} - {\rm i}\,\bar{\Phi}_{\rm T}. \tag{111}$$

We also define a new complex energy–momentum source $\mathcal{M}_\pm := \mathcal{M}_{\rm T} \pm {\rm i}\,\bar{\mathcal{M}}_{\rm T}$ and potential $V_\pm := \tilde{V} \pm \tilde{U}$, where the '±' is relative. Therefore, by taking complex combinations of (108) and (109), we find two new decoupled equations given by

$$\left\{\left[\mathcal{L}_u - 2\Sigma + \frac{7}{3}\theta + (2\phi - 4\mathcal{A})\right]\mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 3\phi - 6\Sigma)\,\mathcal{L}_n - V_+\right\}\Phi_+ = \mathcal{M}_+, \tag{112}$$

$$\left\{\left[\mathcal{L}_u - 2\Sigma + \frac{7}{3}\theta - (2\phi - 4\mathcal{A})\right]\mathcal{L}_u - (\mathcal{L}_n + \mathcal{A} + 3\phi + 6\Sigma)\,\mathcal{L}_n - V_-\right\}\Phi_- = \mathcal{M}_-. \tag{113}$$

It is vital to point out here that, since the covariant differential operators in (112)–(113) are purely real, by taking the real and imaginary parts separately there are actually four real decoupled quantities. It is now of interest to see how $\Phi_\pm$ relates back to the real GEM 2-tensor harmonic amplitudes. The GEM 2-tensors are expanded according to

$$\mathcal{E}_{\mu\nu} = \mathcal{E}_{\rm T} Q_{\mu\nu} + \bar{\mathcal{E}}_{\rm T} \bar{Q}_{\mu\nu} \qquad \text{and} \qquad \mathcal{H}_{\mu\nu} = \mathcal{H}_{\rm T} Q_{\mu\nu} + \bar{\mathcal{H}}_{\rm T} \bar{Q}_{\mu\nu}. \tag{114}$$

Here, the polar perturbations are $\mathcal{E}_{\rm T}$ and $\bar{\mathcal{H}}_{\rm T}$ whereas the axial perturbations are $\bar{\mathcal{E}}_{\rm T}$ and $\mathcal{H}_{\rm T}$. Moreover, a full categorization of all the harmonic amplitudes of the 1+1+2 dependent variables into polar and axial perturbations is presented in [1]. The definition (71) now implies

$$\Phi_{\rm T} := \mathcal{E}_{\rm T} + {\rm i}\,\mathcal{H}_{\rm T} \qquad \text{and} \qquad \bar{\Phi}_T := \bar{\mathcal{E}}_{\rm T} + {\rm i}\,\bar{\mathcal{H}}_{\rm T}, \tag{115}$$

and by subsequently using (111) we find

$$\Phi_+ = (\mathcal{E}_{\rm T} - \bar{\mathcal{H}}_{\rm T}) + {\rm i}(\bar{\mathcal{E}}_{\rm T} + \mathcal{H}_{\rm T}) \qquad \text{and} \qquad \Phi_- = (\mathcal{E}_{\rm T} + \bar{\mathcal{H}}_{\rm T}) - {\rm i}(\bar{\mathcal{E}}_{\rm T} - \mathcal{H}_{\rm T}). \tag{116}$$

Thus, the four precise combinations of the four real GEM 2-tensor harmonic amplitudes which decouple are

$$\text{decoupled polar perturbations:} \quad \{\mathcal{E}_{\mathrm{T}} + \bar{\mathcal{H}}_{\mathrm{T}}, \mathcal{E}_{\mathrm{T}} - \bar{\mathcal{H}}_{\mathrm{T}}\}, \tag{117}$$

$$\text{decoupled axial perturbations:} \quad \{\mathcal{H}_{\mathrm{T}} + \bar{\mathcal{E}}_{\mathrm{T}}, \mathcal{H}_{\mathrm{T}} - \bar{\mathcal{E}}_{\mathrm{T}}\}. \tag{118}$$

Moreover, it is clear that if the four decoupled quantities are known, then simple linear combinations will reveal each of $\mathcal{E}_{\mathrm{T}}$, $\bar{\mathcal{H}}_{\mathrm{T}}$, $\mathcal{H}_{\mathrm{T}}$ and $\bar{\mathcal{E}}_{\mathrm{T}}$.

6. Summary

This paper is the first in a series of papers to discuss (covariant and gauge-invariant) gravitational and energy–momentum perturbations on arbitrary vacuum LRS class II spacetimes. We showed how particular combinations of the first-order GEM quantities decouple at two different levels. The first was a complex tensorial equation governing the complex GEM 2-tensor $\Phi_{\mu\nu}$ (102). The second involved a tensor harmonic expansion of the GEM 2-tensors and resulted in four real equations (112)–(113). Of particular interest is that we have found the precise combinations of the GEM 2-tensor harmonic amplitudes that decouple, and these were separated out into polar and axial perturbations according to (117)–(118). It is also important to note that each individual GEM 2-tensor amplitude on its own does not satisfy a decoupled wave equation. This property was also demonstrated for the Schwarzschild spacetime in [1] where they chose a special frame and derived second-order differential equations for each of $\mathcal{E}_{\mu\nu}$ and $\mathcal{H}_{\mu\nu}$ which were both clearly coupled to the GEM 2-vectors and shear 2-tensors.

The next paper in this series focuses on decoupling the GEM 2-vector amplitudes. In fact, immediate difficulties arise if we attempt to follow the same procedure as presented here to try and derive a decoupled equation governing the 1+1+2 complex GEM 2-vector, Φ_μ. Consider taking the Lie derivative with respect to u^μ of (69), then it follows that you must have an evolution equation for Υ_μ which then implies you need an evolution equation for the first-order quantity $\mathcal{A}_\mu$, for which there is none. Thus, we show that only when the complex GEM 2-vector is combined with other 1+1+2 2-tensors do they decouple. Therefore, we modify the complex GEM system and ultimately a vector harmonic expansion results in another four real decoupled quantities. The following paper will then focus on decoupling the GEM scalar harmonic amplitudes into 3 RW equations for LRS class II spacetimes and when reduced to the Schwarzschild case, 2 of these are indirectly related to the RW equation derived in [1]. It will be here that we present a summary of the 11 decoupled quantities arising from the 1+1+2 complex GEM system. Furthermore, we will show that there are only two dynamical quantities and once known, the remaining 1+1+2 GEM system can be found without further integration. Finally, in the last paper we will show how to use the information calculated from the complex 1+1+2 GEM system to solve the remaining 1+1+2 Ricci identities and consider gravitational radiation applications. Of particular interest is to perturb the background spacetime with a specific first-order energy–momentum distribution. Furthermore, there is a close relationship between the 1+1+2 formalism and the Newman–Penrose (NP) formalism [24], as it is always possible to express the 1+1+2 frame vectors in terms of the NP null vectors. Thus, the following question will be addressed: in LRS class II spacetimes of astrophysical interest, can the 1+1+2 approach lead to genuine new physics that the NP approach fails to achieve? The expectations are promising as there will be a total of 11 decoupled quantities, each giving different information regarding the physics. Moreover, the 1+3 splitting enjoyed significant success for our understanding of cosmological applications, and based on this success, it is perhaps reasonable to also have strong prospects for the 1+1+2 formalism to be successful.

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