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# 1+1+2 electromagnetic perturbations on general LRS spacetimes: Regge–Wheeler and Bardeen–Press equations

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**Abstract**

We use the (covariant and gauge-invariant) 1+1+2 formalism developed by Clarkson and Barrett (2003 *Class. Quanum Grav.* **20** 3855–84), and develop new techniques, to decouple electromagnetic (EM) perturbations on arbitrary locally rotationally symmetric (LRS) spacetimes. Ultimately, we derive three decoupled complex equations governing three complex scalars. One of these is a new Regge–Wheeler (RW) equation generalized for LRS spacetimes, whereas the remaining two are new generalizations of the Bardeen–Press (BP) equations. This is achieved by first using linear algebra techniques to rewrite the first-order Maxwell equations in a new complex 1+1+2 form which is conducive to decoupling. This new complex system immediately yields the generalized RW equation, and furthermore, we also derive a decoupled equation governing a newly defined complex EM 2-vector. Subsequently, a further decomposition of the 1+1+2 formalism into a 1+1+1+1 formalism is developed, allowing us to decompose the complex EM 2-vector, and its governing equations, into spin-weighted scalars, giving rise to the generalized BP equations.

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## 1. Introduction

The 1+3 formalism is well established (for example, see [2–4]), whereby a 4-velocity, $u^\mu$, is defined such that it is both time-like and normalized, $u^\alpha u_\alpha := -1$. Any tensor may then be irreducibly decomposed into several parts; scalars by contracting with the four-velocity, a spatial part (referred to as a 3-tensor) which is projected onto the instantaneous rest-space (or 3-sheet) orthogonal to $u^\mu$, and parts which comprise combinations of both projections and contractions in the time-like direction (also 3-tensors). Recently, Clarkson and Barrett developed a further decomposition of the 1+3 formalism into a 1+1+2 formalism for an analysis of gravitational perturbations of a covariant description of the Schwarzschild spacetime [1]. They introduced a radial vector field, $n^\mu$, defined such that it is space-like and normalized, $n^\alpha n_\alpha := 1$, and it is orthogonal to the time-like vector field, $u^\alpha n_\alpha = 0$. In this way, every 3-tensor may be further irreducibly decomposed into scalars by contracting with the radial vector, a part which is projected onto a 2-sheet (called a 2-tensor) orthogonal to both $u^\mu$ and $n^\mu$, and parts which comprise combinations.

The 1+1+2 formalism is an ideal setting for describing electromagnetic (EM) perturbations on locally rotationally symmetric (LRS) spacetimes [3, 5–7]. LRS spacetimes were first classified by [3, 7] who used an orthonormal tetrad system in order to study dust and general fluid spacetimes. A spacetime is said to be LRS if there exists a preferred spatial direction, and furthermore, any 3-tensor must then lie parallel to this direction; a very simple example is a Schwarzschild black hole where the preferred spatial direction would be parallel to the radial direction. A covariant approach to LRS spacetimes using the 1+3 splitting was later developed in [6]. They employed the 1+3 formalism and introduced a preferred spatial 3-vector, $e_\mu$, defined covariantly by $e_\alpha e^\alpha = 1$ and $e_\alpha u^\alpha = 0$. Therefore, since LRS implies that any other 3-vector, $v_\mu$, must lie parallel to the preferred direction it may be expressed as $v_\mu = \alpha e_\mu$ where $\alpha$ is some scalar function (and similar results hold true for 3-tensors of greater type). Subsequently, the 1+3 equations reduced to partial differential equations involving these scalar quantities only. Thus, the 1+1+2 formalism is ideally suited to study LRS spacetimes by letting the preferred direction be given by the radial vector $n^\mu$ defined in [5]. Under these conditions, all the 2-tensors vanish and only the scalar quantities remain. Furthermore, the 1+1+2 formalism has strong prospects to be as successful to learn about astrophysical LRS spacetimes, as the 1+3 formalism was for understanding cosmological models. EM perturbations via the 1+1+2 formalism were first considered in [5] for LRS class II spacetimes, which comprise a sub-set of LRS spacetimes and this is discussed further in section 3.1.1. In section 2, the necessary background constraint, evolution, propagation and transportation equations are presented for arbitrary LRS spacetimes, including a full description of energy–momentum sources.

In section 3, EM perturbations on arbitrary LRS spacetimes are considered and the corresponding first-order Maxwell equations in 1+1+2 form are reproduced from [8]. Subsequently, we use eigenvalue/eigenvector analysis to reveal that a very natural way to decouple the 1+1+2 Maxwell equations is to construct new complex dependent variables. Consequently, we display Maxwell's equations in a new 1+1+2 complex form which is conducive to decoupling. Then in section 3.1 we derive a Regge–Wheeler (RW) equation [9], generalized towards arbitrary LRS spacetimes for a complex scalar. In section 3.2, we derive a decoupled equation governing a complex 2-tensor. Subsequently, in order to decouple the individual components, we again use linear algebra techniques to best exploit the inherent structure of the equations. Consequently, a further decomposition of the 1+1+2 formalism into a new 1+1+1+1 formalism is developed and the complex 2-tensor $\Phi_\mu$ is irreducibly decomposed into two spin-weighted scalars [10]. Ultimately, we arrive at a generalization of the Bardeen–Press (BP) equations [10–12] for LRS spacetimes. Other studies of EM perturbations include [13, 14] who analyzed four-dimensional charged black holes and showed that specific combinations of the EM and metric perturbation quantities decouple. Furthermore, this has been subsequently generalized by [15, 16] whereby harmonic expansions, as defined in [17–19], were used to find decoupled quantities governing perturbations to $n$ dimensional charged black holes.

Finally, unless otherwise stated, we adhere to the notation employed in [1, 5].

## 2. The background LRS spacetime

There is a set of 1+1+2 scalar quantities describing arbitrary LRS spacetimes (also noted in [5]) which are given by

$$\text{LRS:} \quad \{\mathcal{A}, \theta, \phi, \Sigma, \xi, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, \mathcal{Q}, \Pi, \Lambda\}. \tag{1}$$

Here $\mathcal{A}$ is the radial acceleration of the 4-velocity, $\theta$ and $\phi$ are respectively the expansions of the 3-sheets and 2-sheets, $\Sigma$ is the radial part of the shear of the 3-sheet, $\xi$ is the twisting of the 2-sheet and $\Omega$ is the radial part of the vorticity of the 3-sheet. Also, the radial parts of the gravito-electric and gravito-magnetic tensors are respectively, $\mathcal{E}$ and $\mathcal{H}$. The energy–momentum quantities, mass–energy density, pressure, radial heat flux and radial anisotropic stress are denoted respectively $\mu$, $p$, $\mathcal{Q}$ and $\Pi$, and finally, $\Lambda$ is the cosmological constant. The 2-derivative associated with the 2-sheets is defined as

$$\delta_\mu U_{\nu\ldots\sigma} := N_\mu^\alpha N_\nu{}^\beta \ldots N_\sigma{}^\gamma \nabla_\alpha U_{\beta\ldots\gamma}, \tag{2}$$

where $U_{\nu\ldots\sigma}$ is a 2-tensor and $N_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu - n_\mu n_\nu$ is the projection tensor which projects onto the 2-sheets ($g_{\mu\nu}$ is the 4-metric), and by operating on any LRS scalar will yield zero. The 1+1+2 coupled system which governs these scalars arises from the Ricci identities for the vector fields, $u^\mu$ and $n^\mu$ and the Bianchi identities. They were presented in [1] for the vacuum Schwarzschild case and in [5] for LRS class II spacetimes. Here, we generalize them further for arbitrary LRS spacetimes and the system becomes significantly more complicated. Furthermore, an independent study of these equations has been carried out in a very recent paper [20]. Compared to LRS class II spacetimes, there is a combined total of seven additional evolution, propagation and constraint equations arising which govern the additional scalar quantities. First, the Ricci and Bianchi identities are defined according to

$$Q_{\mu\nu\sigma} := 2\nabla_{[\mu}\nabla_{\nu]}u_\sigma - R_{\mu\nu\sigma\tau}u^\tau = 0, \tag{3}$$

$$R_{\mu\nu\sigma} := 2\nabla_{[\mu}\nabla_{\nu]}n_\sigma - R_{\mu\nu\sigma\tau}n^\tau = 0, \tag{4}$$

$$B_{\nu\sigma\tau} := \nabla^\mu C_{\mu\nu\sigma\tau} - \left[\nabla_{[\sigma}T_{\tau]\nu} + \tfrac{1}{3}g_{\nu[\sigma}\nabla_{\tau]}T\right] = 0, \tag{5}$$

where $C_{\mu\nu\sigma\tau}$ is the Weyl tensor, $R_{\mu\nu\sigma\tau}$ is the Riemann tensor, $T_{\mu\nu}$ the energy–momentum tensor and $\nabla_\mu$ the four-dimensional covariant derivative. The governing equations are then categorized into specific groups according to

Constraint[3]

$$3\xi\Sigma - (2\mathcal{A} - \phi)\Omega - \mathcal{H} = 0 \tag{6}$$

Propagation[4]

$$\hat{\phi} + \tfrac{1}{2}\phi^2 + \left(\Sigma - \tfrac{2}{3}\theta\right)\left(\Sigma + \tfrac{1}{3}\theta\right) + \mathcal{E} - 2\xi^2 = -\tfrac{2}{3}(\mu + \Lambda) - \tfrac{1}{2}\Pi, \tag{7}$$

$$\hat{\Sigma} - \tfrac{3}{2}\hat{\theta} + \tfrac{3}{2}\phi\Sigma + 2\xi\Omega = -\mathcal{Q}, \tag{8}$$

$$\hat{\mathcal{E}} + \tfrac{3}{2}\phi\mathcal{E} - 3\mathcal{H}\Omega = \tfrac{1}{3}\hat{\mu} + \tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\mathcal{Q} - \tfrac{1}{2}\hat{\Pi} - \tfrac{3}{4}\phi\Pi, \tag{9}$$

$$\hat{\mathcal{H}} + \tfrac{3}{2}\phi\mathcal{H} + 3\mathcal{E}\Omega = -\left(\mu + p - \tfrac{1}{2}\Pi\right)\Omega - \mathcal{Q}\xi, \tag{10}$$

$$\hat{\xi} + \phi\xi - \left(\Sigma + \tfrac{1}{3}\theta\right)\Omega = 0, \tag{11}$$

$$\hat{\Omega} - (\mathcal{A} - \phi)\Omega = 0. \tag{12}$$

[3] They are derived as follows: (6) from $\epsilon^{\mu\nu}u^\sigma R_{\mu\nu\sigma} = 0$.

[4] Derived as follows: (7) from $n^\mu N^{\nu\sigma}R_{\mu\nu\sigma} = 0$; (8) from $n^\mu N^{\nu\sigma}Q_{\mu\nu\sigma} = 0$; (9) from $u^\alpha u^\beta n^\gamma B_{\alpha\beta\gamma} = 0$; (10) from $\epsilon^{\beta\gamma}u^\alpha B_{\alpha\beta\gamma} = 0$; (11) from $n^\mu\epsilon^{\nu\sigma}R_{\mu\nu\sigma} = 0$; (12) from $\epsilon^{\mu\nu\sigma}Q_{\mu\nu\sigma} = 0$.

Evolution[5]

$$\dot{\phi} + \left(\Sigma - \tfrac{2}{3}\theta\right)\left(\mathcal{A} - \tfrac{1}{2}\phi\right) - 2\xi\Omega = \mathcal{Q}, \tag{13}$$

$$\dot{\Sigma} - \tfrac{2}{3}\dot{\theta} - \tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)^2 + \mathcal{A}\phi + \mathcal{E} + 2\Omega^2 = \tfrac{1}{3}(\mu + 3p - 2\Lambda) + \tfrac{1}{2}\Pi, \tag{14}$$

$$\dot{\mathcal{E}} - \tfrac{3}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\mathcal{E} - 3\mathcal{H}\xi = \tfrac{1}{3}\dot{\mu} - \tfrac{1}{2}\dot{\Pi} + \tfrac{1}{4}\left(\Sigma - \tfrac{2}{3}\theta\right)\Pi + \tfrac{1}{2}\phi\mathcal{Q} - \tfrac{1}{2}(\mu + p)\left(\Sigma - \tfrac{2}{3}\theta\right), \tag{15}$$

$$\dot{\mathcal{H}} - \tfrac{3}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\mathcal{H} + 3\mathcal{E}\xi = \mathcal{Q}\Omega + \tfrac{3}{2}\Pi\xi, \tag{16}$$

$$\dot{\xi} - 2\left(\Sigma - \tfrac{1}{6}\theta\right)\xi = 0, \tag{17}$$

$$\dot{\Omega} - \left(\Sigma - \tfrac{2}{3}\theta\right)\Omega - \mathcal{A}\xi = 0. \tag{18}$$

Transportation[6]

$$\hat{\mathcal{A}} + (\mathcal{A} + \phi)\mathcal{A} - \dot{\theta} - \tfrac{1}{3}\theta^2 - \tfrac{3}{2}\Sigma^2 + 2\Omega^2 = \tfrac{1}{2}(\mu + 3p - 2\Lambda), \tag{19}$$

$$\dot{\mu} + \theta\mu + \hat{\mathcal{Q}} + (2\mathcal{A} + \phi)\mathcal{Q} + \theta p + \tfrac{3}{2}\Sigma\Pi = 0, \tag{20}$$

$$\dot{\mathcal{Q}} + \left(\Sigma + \tfrac{4}{3}\theta\right)\mathcal{Q} + \hat{p} + \mathcal{A}p + \hat{\Pi} + \left(\mathcal{A} + \tfrac{3}{2}\phi\right)\Pi + \mu\mathcal{A} = 0. \tag{21}$$

Moreover, it is possible to derive a transport equation for $\mathcal{A}$ by substituting the constraint (6) into (16) and subsequently eliminating all other dot derivatives using (13), (14) and (17)–(19) to achieve

$$\Omega\dot{\mathcal{A}} - \xi\hat{\mathcal{A}} = 0. \tag{22}$$

Here, the 'dot' derivative is defined as $\dot{X}_{\mu\ldots\nu} := u^\alpha\nabla_\alpha X_{\mu\ldots\nu}$ where $X_{\mu\ldots\nu}$ represents any quantity. The 'hat' derivative is defined as $\hat{W}_{\mu\ldots\nu} := n^\alpha D_\alpha W_{\mu\ldots\nu}$, where $W_{\mu\ldots\nu}$ represents a 3-tensor and $D_\mu$ is the derivative associated with the 3-sheets, defined by

$$D_\mu W_{\nu\ldots\sigma} := h_\mu{}^\alpha h_\nu{}^\beta \ldots h_\sigma{}^\gamma \nabla_\alpha W_{\beta\ldots\gamma}, \tag{23}$$

where the tensor which projects onto the 3-sheets is defined as $h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$.

## 3. EM perturbations on LRS spacetimes

We now consider first-order EM perturbations to arbitrary LRS background spacetimes defined by (1) and (6)–(21). The EM perturbations ($E_\mu$ and $B_\mu$) and the current 3-vector ($J_\mu$) are covariant and are considered to be gauge-invariant according to the Sachs–Stewart–Walker lemma [21, 22]. They are irreducibly split into 1+1+2 form according to

$$E_\mu = \mathscr{E} n_\mu + \mathscr{E}_\mu, \qquad B_\mu = \mathscr{B} n_\mu + \mathscr{B}_\mu \qquad \text{and} \qquad J_\mu = \mathscr{J} n_\mu + \mathscr{J}_\mu. \tag{24}$$

The fully nonlinear Maxwell equations were previously presented in 1+1+2 form [8]. The corresponding, covariant and gauge-invariant, first-order equations become

$$\hat{\mathscr{B}} + \phi\mathscr{B} + \delta^\alpha\mathscr{B}_\alpha + 2\Omega\mathscr{E} = 0, \tag{25}$$

$$\hat{\mathscr{E}} + \phi\mathscr{E} + \delta^\alpha\mathscr{E}_\alpha - 2\Omega\mathscr{B} = \rho_e, \tag{26}$$

$$\dot{\mathscr{B}} - \left(\Sigma - \tfrac{2}{3}\theta\right)\mathscr{B} + \epsilon^{\alpha\beta}\delta_\alpha\mathscr{E}_\beta + 2\mathscr{E}\xi = 0, \tag{27}$$

[5] Derived as follows: (13) from $u^\mu N^{\nu\sigma} R_{\mu\nu\sigma} = 0$; (14) from $u^\mu N^{\nu\sigma} Q_{\mu\nu\sigma} = 0$; (15) from $u^\alpha n^\beta n^\gamma B_{\beta\gamma\alpha} = 0$ and (20); (16) from $\epsilon^{\alpha\beta} n^\gamma B_{\gamma\alpha\beta} = 0$; (17) from $u^\mu \epsilon^{\nu\sigma} R_{\mu\nu\sigma} = 0$ and (6); (18) from $u^\mu \epsilon^{\nu\sigma} Q_{\mu\nu\sigma} = 0$.

[6] Derived as follows: (19) from $u^\mu n^\nu u^\sigma R_{\mu\nu\sigma}$.

$$\dot{\mathscr{E}} - \left(\Sigma - \tfrac{2}{3}\theta\right)\mathscr{E} - \epsilon^{\alpha\beta}\delta_\alpha\mathscr{B}_\beta - 2\mathscr{B}\xi = -\mathscr{J}, \tag{28}$$

$$\dot{\mathscr{B}}_{\bar\mu} + \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\theta\right)\mathscr{B}_\mu - \epsilon_\mu{}^\alpha\left[\hat{\mathscr{E}}_\alpha + \left(\mathcal{A} + \tfrac{1}{2}\phi\right)\mathscr{E}_\alpha\right] + \epsilon_\mu{}^\alpha\delta_\alpha\mathscr{E} + \Omega\epsilon_\mu{}^\alpha\mathscr{B}_\alpha + \xi\mathscr{E}_\mu = 0, \tag{29}$$

$$\dot{\mathscr{E}}_{\bar\mu} + \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\theta\right)\mathscr{E}_\mu + \epsilon_\mu{}^\alpha\left[\hat{\mathscr{B}}_\alpha + \left(\mathcal{A} + \tfrac{1}{2}\phi\right)\mathscr{B}_\alpha\right] - \epsilon_\mu{}^\alpha\delta_\alpha\mathscr{B} + \Omega\epsilon_\mu{}^\alpha\mathscr{E}_\alpha - \xi\mathscr{B}_\mu = -\mathscr{J}_\mu, \tag{30}$$

where in accord with standard notation, a 'bar' over an index implies that the index has been projected onto the 2-sheets. Also, $\rho_e$ is the electric charge density and $\epsilon_{\mu\nu}$ is the anti-symmetric pseudo-2-tensor defined with respect to the four-dimensional Levi-Civita pseudo-tensor according to $\epsilon_{\mu\nu} := \epsilon_{\sigma\mu\nu\tau}u^\sigma n^\tau$.

It has long been established that by constructing a complex combination of the EM fields, the system of equations is greatly simplified [23]. This is due to the inherent structure of Maxwell's equations, and this is also true for the fully nonlinear equations. They are invariant (in the absence of sources) under the simultaneous transformation $E_\mu \to B_\mu$ and $B_\mu \to -E_\mu$, which corresponds to $\{\mathscr{E} \to \mathscr{B}, \mathscr{E}_\mu \to \mathscr{B}_\mu\}$ and $\{\mathscr{B} \to -\mathscr{E}, \mathscr{B}_\mu \to -\mathscr{E}_\mu\}$. Therefore, the appendix uses linear algebra techniques to show that a natural decoupling of the equations is achieved by choosing new dynamical complex quantities according to

$$\Phi := \mathscr{E} + \mathrm{i}\,\mathscr{B} \qquad \text{and} \qquad \Phi_\mu := \mathscr{E}_\mu + \mathrm{i}\,\mathscr{B}_\mu, \tag{31}$$

where i is the complex number[7]. Thus, without loss of generality, the six real Maxwell equations (26)–(30) are expressed in a new 1+1+2 complex form,

$$\hat\Phi + \phi\Phi + \delta^\alpha\Phi_\alpha + \mathrm{i}\,2\Omega\Phi = \rho_e, \tag{32}$$

$$\dot\Phi - \left(\Sigma - \tfrac{2}{3}\theta\right)\Phi + \mathrm{i}\,\epsilon^{\alpha\beta}\delta_\alpha\Phi_\beta + \mathrm{i}\,2\xi\Phi = -\mathscr{J}, \tag{33}$$

$$\dot\Phi_{\bar\mu} + \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\theta\right)\Phi_\mu - \mathrm{i}\,\epsilon_\mu{}^\alpha\left[\hat\Phi_\alpha + \left(\mathcal{A} + \tfrac{1}{2}\phi\right)\Phi_\alpha\right] + \mathrm{i}\,\epsilon_\mu{}^\alpha\delta_\alpha\Phi + \Omega\epsilon_\mu{}^\alpha\Phi_\alpha + \mathrm{i}\,\xi\Phi_\mu = -\mathscr{J}_\mu. \tag{34}$$

Before proceeding with decoupling the 1+1+2 complex system, we write down the commutation relationships between the various derivatives defined throughout. These are very important for the forthcoming analysis and furthermore, it is also vital to perform an integrability check with each and every equation. For any first-order scalar $\Psi$, they were presented previously in [1]

$$\hat{\dot\Psi} - \left(\Sigma + \tfrac{1}{3}\theta\right)\hat\Psi - \dot{\hat\Psi} + \mathcal{A}\dot\Psi = 0, \tag{35}$$

$$\delta_\mu\dot\Psi - (\delta_{\bar\mu}\Psi)^{\cdot} + \tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\delta_\mu\Psi - \Omega\epsilon_\mu{}^\alpha\delta_\alpha\Psi = 0, \tag{36}$$

$$\delta_\mu\hat\Psi - (\delta_{\bar\mu}\Psi)^{\wedge} - \tfrac{1}{2}\phi\delta_\mu\Psi - \xi\epsilon_\mu{}^\alpha\delta_\alpha\Psi = 0, \tag{37}$$

$$\delta_{[\mu}\delta_{\nu]}\Psi - \epsilon_{\mu\nu}(\Omega\dot\Psi - \xi\hat\Psi) = 0. \tag{38}$$

For a first-order 2-vector, $\Psi_\mu$, they were given in [5] for LRS class II spacetimes, and here they are generalized for arbitrary LRS spacetimes (and they may also be found in the very recent independent study of Clarkson [20]),

$$\hat{\dot\Psi}_{\bar\mu} - \left(\Sigma + \tfrac{1}{3}\theta\right)\hat\Psi_{\bar\mu} - \dot{\hat\Psi}_{\bar\mu} + \mathcal{A}\dot\Psi_{\bar\mu} - \mathcal{H}\epsilon_\mu{}^\alpha\Phi_\alpha = 0, \tag{39}$$

$$\delta_\mu\dot\Psi_\nu - (\delta_{\bar\mu}\Psi_{\bar\nu})^{\cdot} + \tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\delta_\mu\Psi_\nu - \Omega\epsilon_\mu{}^\alpha\delta_\alpha\Psi_\nu = 0, \tag{40}$$

[7] It is also possible to alternatively choose the complex conjugates, $\Phi^* := \mathscr{E} - \mathrm{i}\,\mathscr{B}$ and $\Phi^*_\mu := \mathscr{E}_\mu - \mathrm{i}\,\mathscr{B}_\mu$. Furthermore, any equations governing $\Phi^*$ and $\Phi^*_\mu$ may be found by simply taking the complex conjugate of the equations governing $\Phi$ and $\Phi_\mu$.

$$\delta_\mu \hat{\Psi}_\nu - (\delta_{\bar{\mu}} \Psi_{\bar{\nu}})\hat{} - \tfrac{1}{2}\phi \delta_\mu \Psi_\nu - \xi \epsilon_\mu{}^\alpha \delta_\alpha \Psi_\nu = 0, \tag{41}$$

$$\delta_{[\mu}\delta_{\nu]}\Psi_\sigma + K\Psi_{[\mu} N_{\nu]\sigma} - \epsilon_{\mu\nu}\big(\Omega\dot{\Psi}_{\bar{\sigma}} - \xi\hat{\Psi}_{\bar{\sigma}}\big) = 0. \tag{42}$$

Here, the scalar function $K$ has been defined as

$$K := \tfrac{1}{3}(\mu + \Lambda) - \mathcal{E} - \tfrac{1}{2}\Pi + \tfrac{1}{4}\phi^2 - \tfrac{1}{4}\big(\Sigma - \tfrac{2}{3}\theta\big)^2 + \xi^2 - \Omega^2, \tag{43}$$

and this is a natural generalization of the Gaussian curvature scalar defined in [5] for LRS class II spacetimes. In the LRS class II case where $\xi = \Omega = \mathcal{H} = 0$, the sheets mesh to form surfaces for which the Gaussian curvature then has its standard definition [5].

*3.1. Regge–Wheeler equation for LRS spacetimes*

The (gauge-invariant and covariant) decoupled equation governing $\Phi$ is derived by taking the 'dot' derivative of (33) and it is important to use the Ricci/Bianchi identities (7)–(21), the scalar function $K$ (43), and the commutation relationships for the various derivatives (35)–(42). It is also necessary to substitute (32)–(34) for further simplifications and after some arduous manipulation, we arrive at

$$\ddot{\Phi} - \big(\Sigma - \tfrac{5}{3}\theta - \mathrm{i}\,2\xi\big)\dot{\Phi} - \hat{\hat{\Phi}} - (\mathcal{A} + 2\phi + \mathrm{i}\,2\Omega)\hat{\Phi} - V\Phi = \mathcal{S}. \tag{44}$$

The potential and energy–momentum source have been defined,

$$V := \delta^2 + 2K - \mu + p + \Pi - 2\Lambda + \mathrm{i}\,4\big[\Omega\mathcal{A} - \xi\big(\Sigma + \tfrac{1}{3}\theta\big)\big], \tag{45}$$

$$\mathcal{S} := -\hat{\rho}_e - (\phi + \mathcal{A})\rho_e - \dot{\mathscr{J}} - \theta\mathscr{J} + \mathrm{i}\,\epsilon^{\alpha\beta}\delta_\alpha \mathscr{J}_\beta, \tag{46}$$

where the 2-Laplacian is $\delta^2 := \delta^\alpha\delta_\alpha$. This is a new complex RW equation generalized for EM perturbations on arbitrary LRS spacetimes, and this generalizes the RW equation derived in [5] for LRS class II spacetimes.

By inspecting (44), this clearly demonstrates that for arbitrary LRS spacetimes, the complex EM scalar, $\Phi$, decouples from the complex EM 2-vector, $\Phi_\mu$. It also indicates that the radial electric field ($\mathscr{E}$) and radial magnetic field ($\mathscr{B}$) do not decouple from each other and instead they must be treated as a single complex radial electromagnetic field ($\Phi$). We show in the next section that further decoupling can be achieved in specific sub-cases.

It is also noted that it is convenient to introduce a scalar function, $r$, according to

$$\frac{\hat{r}}{r} = \frac{1}{2}\phi, \qquad \frac{\dot{r}}{r} = -\frac{1}{2}\left(\Sigma - \frac{2}{3}\theta\right) \qquad \text{and} \qquad \delta_\mu r = 0, \tag{47}$$

as defined in [1]. In this way a new scaled variable may be chosen, $\Upsilon = r^2\Phi$, such that the scaled RW equation becomes

$$\ddot{\Upsilon} + \big(\Sigma + \tfrac{1}{3}\theta + \mathrm{i}\,2\xi\big)\dot{\Upsilon} - \hat{\hat{\Upsilon}} - (\mathcal{A} + \mathrm{i}\,2\Omega)\hat{\Upsilon} - U\Upsilon = r^2\mathcal{S} \tag{48}$$

and the primary advantage is that the potential scales to a much simpler form,

$$U := \delta^2 - \mathrm{i}\,2[\Omega(\phi - 2\mathcal{A}) + 3\xi\Sigma]. \tag{49}$$

*3.1.1. LRS class II spacetimes.* A closer inspection of the coefficients in the complex RW equation (44) reveals that the imaginary components are always associated with either $\xi$ or $\Omega$. Therefore, further decoupling is achieved in the case when they vanish and by (6) this implies $\mathcal{H}$ will also vanish, and this is precisely LRS class II defined by

$$\text{LRS class II:} \quad \{\mathcal{A}, \theta, \phi, \Sigma, \mathcal{E}, \mu, p, \mathcal{Q}, \Pi, \Lambda\}. \tag{50}$$

Thus, for LRS class II the complex RW equation (44) reduces to

$$\ddot{\Phi} - \left(\Sigma - \tfrac{5}{3}\theta\right)\dot{\Phi} - \hat{\hat{\Phi}} - (\mathcal{A} + 2\phi)\hat{\Phi} - V\Phi = \mathcal{S}, \tag{51}$$

where the potential is now

$$V := \delta^2 + 2K - \mu + p + \Pi - 2\Lambda, \tag{52}$$

which is a remarkably simple form; for example, in vacuum spacetimes whereby the cosmological constant vanishes, the potential is purely in terms of the Gaussian curvature of the 2-sheets (and the 2-Laplacian). Furthermore, the scaled LRS class II RW equation reduces to

$$\ddot{\Upsilon} + \left(\Sigma + \tfrac{1}{3}\theta\right)\dot{\Upsilon} - \hat{\hat{\Upsilon}} - \mathcal{A}\hat{\Upsilon} - \delta^2\Upsilon = r^2\mathcal{S}, \tag{53}$$

and the potential is purely in terms of the 2-Laplacian only.

Now since all differential operators (along with their coefficients) acting on $\Phi$ in (51) are purely real, there are two independent decoupled equations here; one for each of the real and imaginary components of $\Phi$ (i.e. $\mathscr{E}$ and $\mathscr{B}$),

$$\ddot{\mathscr{E}} - \left(\Sigma - \tfrac{5}{3}\theta\right)\dot{\mathscr{E}} - \hat{\hat{\mathscr{E}}} - (\mathcal{A} + 2\phi)\hat{\mathscr{E}} - V\mathscr{E} = -\hat{\rho}_e - (\phi + \mathcal{A})\rho_e - \dot{\mathscr{J}} - \theta\,\mathscr{J}, \tag{54}$$

$$\ddot{\mathscr{B}} - \left(\Sigma - \tfrac{5}{3}\theta\right)\dot{\mathscr{B}} - \hat{\hat{\mathscr{B}}} - (\mathcal{A} + 2\phi)\hat{\mathscr{B}} - V\mathscr{B} = \epsilon^{\alpha\beta}\delta_\alpha\mathscr{J}_\beta \tag{55}$$

and these correspond to those derived in [5]. However, these equations correct an error that resides in the potential of the work presented by [5], for which we now elucidate. For better comparison with [5], we can substitute (43) into the potential (52) to reveal,

$$V = \delta^2 + \tfrac{1}{2}\phi^2 - 2\mathcal{E} + \left(\tfrac{2}{9}\theta - \tfrac{1}{3}\Sigma\right)\left(\tfrac{3}{2}\Sigma - \theta\right) - \tfrac{1}{3}(\mu - 3p + 4\Lambda). \tag{56}$$

We will denote their incorrect potential as $V_{BC}$ and reproduce this from [5],

$$V_{BC} = \delta^2 + \tfrac{1}{2}\phi^2 - 2\mathcal{E} \underbrace{+\left(\tfrac{1}{3}\theta + \Sigma\right)\left(\tfrac{3}{2}\Sigma - \theta\right)}_{\text{incorrect term}} - \tfrac{1}{3}(\mu - 3p + 4\Lambda). \tag{57}$$

However, it is strongly emphasized here that this in no way affects the way in which the equations decouple in [5]. Furthermore, [5] presents an informative and interesting analysis of various applications for which the 'incorrect term' vanishes, and thus those results remain intact.

### *3.2. Bardeen–Press equations for LRS spacetimes*

We now show that since we are exploiting the inherent structure of the equations, we can derive a new decoupled equation for the complex EM 2-vector, $\Phi_\mu$. The derivation is similar to how the generalized RW equation was constructed. First take the dot derivative of (34) and use (7)–(42), and substitute (32)–(34), to simplify further. Ultimately, we find a decoupled, covariant and gauge-invariant, equation given by

$$\begin{aligned}\ddot{\Phi}_{\bar{\mu}} &- \left(\Sigma - \tfrac{5}{3}\theta - \mathrm{i}\,2\xi\right)\dot{\Phi}_{\bar{\mu}} - \hat{\hat{\Phi}}_{\bar{\mu}} - (\mathcal{A} + 2\phi + \mathrm{i}\,2\Omega)\hat{\Phi}_{\bar{\mu}} - V_{(1)}\Phi_\mu \\ &- \mathrm{i}\,\epsilon_\mu{}^\alpha[(2\mathcal{A} - \phi + \mathrm{i}\,2\Omega)\dot{\Phi}_\alpha - (3\Sigma + \mathrm{i}\,2\xi)\hat{\Phi}_\alpha - V_{(2)}\Phi_\alpha] = \mathcal{S}_\mu,\end{aligned} \tag{58}$$

where two terms related to the potentials have been defined,

$$\begin{aligned}V_{(1)} := {} & \delta^2 + \mathcal{E} + \tfrac{1}{4}\phi^2 - \mathcal{A}^2 + \phi\mathcal{A} + \tfrac{7}{4}\Sigma^2 - \tfrac{2}{9}\theta^2 + \tfrac{2}{3}\theta\Sigma - \tfrac{1}{3}\mu + p - \tfrac{4}{3}\Lambda + \xi^2 - \Omega^2 \\ & + \mathrm{i}\left[\Omega(2\mathcal{A} + \phi) - \xi\left(\Sigma + \tfrac{4}{3}\theta\right) - \mathcal{H}\right],\end{aligned} \tag{59}$$

$$V_{(2)} := -\dot{\mathcal{A}} + \hat{\theta} + \tfrac{2}{3}\theta(\phi - 2\mathcal{A}) + \tfrac{1}{2}\Sigma(\phi + 4\mathcal{A}) - 2\xi\Omega - \mathcal{Q} + \mathrm{i}\left[2\Omega\left(\Sigma + \tfrac{1}{3}\theta\right) - 2\xi\mathcal{A}\right], \quad (60)$$

and the energy–momentum source

$$\mathcal{S}_\mu := -\dot{\mathscr{J}}_{\bar{\mu}} + \tfrac{3}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\mathscr{J}_\mu - \delta_\mu\rho_e + \Omega\epsilon_\mu{}^\alpha\mathscr{J}_\alpha + \mathrm{i}\,\epsilon_\mu{}^\alpha\left(\delta_\alpha\mathscr{J} - \hat{\mathscr{J}}_\alpha - \tfrac{3}{2}\phi\mathscr{J}_\alpha\right) - \mathrm{i}\,\xi\,\mathscr{J}_\mu. \quad (61)$$

It is now clear that the complex EM 2-tensor also decouples from the complex EM scalar $\Phi$. Thus analogous to the radial case, the electric 2-vector ($\mathscr{E}_\mu$) and the magnetic 2-vector ($\mathscr{B}_\mu$) do not decouple from each other; however, they combine to form a single complex electromagnetic 2-vector ($\Phi_\mu$). It is difficult to eliminate all the derivative terms in part of the potential, $V_{(2)}$, given by (60). Although, it is possible to simplify it slightly by using the transportation equation for $\mathcal{A}$ (22) to eliminate $\dot{\mathcal{A}}$. However, this would be a restricted case as careful consideration would be needed to ensure that the ratio, $\xi\hat{\mathcal{A}}/\Omega$, is well defined.

In the next section we show how to further decompose (58) to find two new BP equations generalized for EM perturbations to LRS spacetimes.

*3.2.1. 1+1+1+1 decomposition.* In order to decouple the two components residing in (58), we consider a further projection along two more vectors. The natural decoupling methodology in the appendix is employed again, and this allows these vectors to be a complex-conjugate pair ($m^\mu, m^{*\mu}$) which satisfies the following relationships:

$$m^{*\alpha}m_\alpha = 1, \qquad m^\alpha m_\alpha = 0, \qquad m^{*\alpha}m^*_\alpha = 0, \qquad N^{\mu\nu} = 2m^{(\mu}m^{*\nu)}, \quad (62)$$

where $N_{\mu\nu}$ is the projection tensor for the 2-sheets and these complex-conjugate vectors are orthogonal to both $u^\mu$ and $n^\mu$. Consider the arbitrary 2-vector, $\Psi_\mu$, and 2-tensor, $\Psi_{\mu\nu}$, then by subsequently using (62), a new irreducible decomposition into 1+1+1+1 form is given by,

$$\Psi_\mu = (\Psi_\alpha m^\alpha)m^*_\mu + (\Psi_\alpha m^{*\alpha})m_\mu, \quad (63)$$

$$\Psi_{\mu\nu} = (\Psi_{\alpha\beta}m^{*\alpha}m^{*\beta})m_\mu m_\nu + (\Psi_{\alpha\beta}m^\alpha m^\beta)m^*_\mu m^*_\nu + 2m_{(\mu}m^*_{\nu)}(\Psi_{\alpha\beta}m^{*(\alpha}m^{\beta)}) + 2m_{[\mu}m^*_{\nu]}(\Psi_{\alpha\beta}m^{*[\alpha}m^{\beta]}), \quad (64)$$

where the individual components are scalars with a specific spin-weight [10] that has a standard definition as follows. Let the complex vector, $m^\mu$, undergo a transformation on the 2-sheet according to $m^\mu \to \mathcal{C}\mathcal{C}^{*-1}m^\mu$ where $\mathcal{C}$ is an arbitrary complex scalar field and $\mathcal{C}^*$ its complex conjugate. Then any quantity, $\zeta_{\mu\ldots\nu}{}^{\sigma\ldots\tau}$, which has a corresponding transformation of

$$\zeta_{\mu\ldots\nu}{}^{\sigma\ldots\tau} \to \mathcal{C}^p\mathcal{C}^{*q}\zeta_{\mu\ldots\nu}{}^{\sigma\ldots\tau}, \quad (65)$$

is said to have a spin-weight $s$ defined,

$$s := \tfrac{1}{2}(p - q). \quad (66)$$

We now derive the new and important quantities which arise from the further decomposition of the 1+1+2 formalism into the 1+1+1+1 formalism. All the subsequent equations will naturally occur in complex-conjugate pairs. However, we only display one of the pairs and note that the other is found by taking the complex conjugate. Thus we have

$$\delta_\mu m_\nu = -m_\mu m_\nu m^{*\alpha}\chi_\alpha - \tfrac{1}{2}N_{\mu\nu}m^\alpha\chi_\alpha - \mathrm{i}\,\tfrac{1}{2}\sigma m^\alpha\chi_\alpha\epsilon_{\mu\nu}, \quad (67)$$

where

$$\chi_\mu := m^\alpha\delta_\mu m^*_\alpha = -\chi^*_\mu \quad (68)$$

is purely imaginary and has zero spin-weight, and

$$\sigma := \mathrm{i}\,m_\alpha m^*_\beta\epsilon^{\alpha\beta} \qquad \text{from which it follows} \qquad \sigma^2 = 1, \quad (69)$$

therefore, $\sigma$ has zero spin-weight and is purely real. Thus, (67) depends only on the complex-conjugate pair $(m^{\alpha}\chi_{\alpha}, m^{*\alpha}\chi^{*}_{\alpha})$ which have a spin weight of 1 and $-1$ respectively. We also have a constraint and a relationship for the divergence, which are respectively

$$(\delta^{\alpha} + \chi^{\alpha})m_{\alpha} = 0 \qquad \text{and} \qquad \epsilon^{\alpha\beta}\delta_{\alpha}m_{\beta} = \mathrm{i}\,\sigma\delta^{\alpha}m_{\alpha}, \tag{70}$$

and finally, by using (64), the Levi-Civita pseudo-2-tensor is decomposed as

$$\epsilon_{\mu\nu} = \mathrm{i}\,2\sigma m_{[\mu}m^{*}_{\nu]}. \tag{71}$$

*3.2.2. Bardeen–Press equations for spin weighted scalars.* We now have developed the necessary mathematical tools to irreducibly decompose the complex 2-vector as

$$\Phi_{\mu} = \mathcal{M}_{\oplus}m^{*}_{\mu} + \mathcal{M}_{\otimes}m_{\mu}, \tag{72}$$

where $\mathcal{M}_{\oplus} := m^{\alpha}\Phi_{\alpha}$ has a spin-weight of $s = 1$ and $\mathcal{M}_{\otimes} := m^{*\alpha}\Phi_{\alpha}$ has a spin-weight of $s = -1$. Finally, by substituting (72) into (58) and contracting separately with $m^{\mu}$ and $m^{*\mu}$, we find the components naturally decouple into two spin-weighted equations of the form,

$$\begin{aligned}\ddot{\mathcal{M}}_{\oplus} &+ \big[2\gamma - \Sigma + \tfrac{5}{3}\theta + \mathrm{i}\,2\xi - s\sigma(2\mathcal{A} - \phi - \mathrm{i}\,2\Omega)\big]\dot{\mathcal{M}}_{\oplus} - \hat{\hat{\mathcal{M}}}_{\oplus}\\ &- [2\lambda + \mathcal{A} + 2\phi - s\sigma(3\Sigma - \mathrm{i}\,2\xi)]\hat{\mathcal{M}}_{\oplus} - 2\chi^{\alpha}\delta_{\alpha}\mathcal{M}_{\oplus} - V_{B}\mathcal{M}_{\oplus} = S_{\oplus},\end{aligned} \tag{73}$$

$$\begin{aligned}\ddot{\mathcal{M}}_{\otimes} &+ \big[2\gamma^{*} - \Sigma + \tfrac{5}{3}\theta + \mathrm{i}\,2\xi - s\sigma(2\mathcal{A} - \phi - \mathrm{i}\,2\Omega)\big]\dot{\mathcal{M}}_{\otimes} - \hat{\hat{\mathcal{M}}}_{\otimes}\\ &- [2\lambda^{*} + \mathcal{A} + 2\phi - s\sigma(3\Sigma - \mathrm{i}\,2\xi)]\hat{\mathcal{M}}_{\otimes} - 2\chi^{*\alpha}\delta_{\alpha}\mathcal{M}_{\otimes} - V_{P}\mathcal{M}_{\otimes} = S_{\otimes}.\end{aligned} \tag{74}$$

where the energy–momentum source has similarly been decomposed as $S_{\mu} := S_{\oplus}m^{*}_{\mu} + S_{\otimes}m_{\mu}$. Furthermore, some new definitions involving various combinations of $m^{\mu}$ and $m^{*\mu}$ are,

$$\begin{aligned}&\gamma := m^{\alpha}\dot{m}^{*}_{\alpha}, \qquad \lambda := m^{\alpha}\hat{m}^{*}_{\alpha}, \qquad \chi_{\mu} := m^{\alpha}\delta_{\mu}m^{*}_{\alpha}\\ &\text{and} \qquad \chi := (\delta^{\alpha}m^{\beta})(\delta_{\alpha}m^{*}_{\beta}),\end{aligned} \tag{75}$$

and the terms related to the potentials are now

$$\begin{aligned}V_{B} := &-\dot{\gamma} - \gamma\big[\gamma - \Sigma + \tfrac{5}{3}\theta - s\sigma(2\mathcal{A} - \phi)\big] + \hat{\lambda} + \lambda(\lambda + \mathcal{A} + 2\phi - 3s\sigma\Sigma)\\ &- \chi + \delta^{\alpha}\chi_{\alpha} + V_{(1)} - \sigma V_{(2)},\end{aligned} \tag{76}$$

$$\begin{aligned}V_{P} := &-\dot{\gamma}^{*} - \gamma^{*}\big[\gamma^{*} - \Sigma + \tfrac{5}{3}\theta - s\sigma(2\mathcal{A} - \phi)\big] + \hat{\lambda}^{*} + \lambda^{*}(\lambda^{*} + \mathcal{A} + 2\phi - 3s\sigma\Sigma)\\ &- \bar{\chi} + \delta^{\alpha}\chi^{*}_{\alpha} + V_{(1)} + \sigma V_{(2)}.\end{aligned} \tag{77}$$

The decoupled equations, (73)–(74), are new generalizations of the BP equations for arbitrary LRS spacetimes.

To reduce the generalized BP equations for a particular application involving a specific LRS metric, one can first determine the two frame vectors $u^{\mu}$ and $n^{\mu}$ and the two complex-conjugate vectors $m^{\mu}$ and $m^{*\mu}$. With these four vectors known, it is then possible to calculate every other required quantity, including all background LRS scalars, from original definitions (see [1] for the LRS class II scalars arising from the Schwarzschild metric). The BP equations were checked for accuracy by using this approach to write them in coordinate form and then comparing them with those derived using the Newman–Penrose (NP) formalism [24] also expressed in coordinate form according to [12]. They were cross-checked using Maple 9.5 and they correspond precisely. In order to achieve this, the NP null vectors were expressed in terms of the 1+1+2 frame vectors according to $k_{\mu} = (u_{\mu} + n_{\mu})f/\sqrt{2}$ and $\ell_{\mu} = (u_{\mu} - n_{\mu})/(f\sqrt{2})$, where $f$ is a scalar function, and the 1+1+1+1 complex-conjugate vectors were chosen to align with the NP complex-conjugate vectors.

## 4. Summary and conclusions

We have successfully decoupled (gauge-invariant and covariant) EM perturbations on arbitrary LRS spacetimes. We used an eigenvector/eigenvalue analysis to take advantage of the inherent mathematical characteristics of Maxwell's equations and express them in a complex 1+1+2 form that facilitates decoupling. This new complex system was then used to demonstrate the decoupling of the complex EM scalar ($\Phi$) and the complex EM 2-vector ($\Phi_\mu$). The governing equation for $\Phi$ is a RW equation generalized towards arbitrary LRS spacetimes. Furthermore, we also derived a new decoupled equation governing the complex EM 2-vector. We then developed a further decomposition of the 1+1+2 formalism into a 1+1+1+1 formalism and ultimately derived a pair of decoupled spin-weighted scalars. The governing equations are the BP equations generalized for arbitrary LRS spacetimes. Finally, we also noted that additional decoupling could be achieved between the EM scalars, $\mathscr{E}$ and $\mathscr{B}$, by reducing the RW equation to the LRS class II sub-case.

This process presented here is also highly useful as a mathematical guide for decoupling the analogous case of gravitational perturbations to LRS spacetimes using the 1+1+2 gravito-electromagnetic (GEM) formalism. We have already shown how to decouple complex GEM spin-weighted scalars for the covariant Schwarzschild case [25] and we will show in a future paper that this can be extended for general LRS spacetimes.

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## Appendix. Linear algebra: decoupling systems of differential equations

Consider the system given by,

$$L_1 E + L_2 B = 0 \qquad \text{and} \qquad L_1 B - L_2 E = 0, \tag{A.1}$$

where $L_1$ and $L_2$ represent differential operators and $E$ and $B$ are any scalar fields. This system has the property that it is invariant under the simultaneous transformation of $E \to B$ and $B \to -E$. This system can be expressed in a matrix form as

$$\begin{pmatrix} L_1 E \\ L_1 B \end{pmatrix} + M \begin{pmatrix} L_2 E \\ L_2 B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \text{where} \qquad M := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{A.2}$$

is the matrix responsible for coupling $E$ to $B$. $M$ can be written in terms of its eigenvalues, diag($D$), and corresponding eigenvectors, col($P$), according to $M = PDP^{-1}$. Therefore, since $D$ is diagonal, it is clear that by multiplying (A.2) by $-2\mathrm{i}\,P^{-1}$ results in the decoupled system,

$$L_1(E + \mathrm{i}\,B) + \mathrm{i}\,L_2(E + \mathrm{i}\,B) = 0 \qquad \text{and} \qquad L_1(E - \mathrm{i}\,B) - \mathrm{i}\,L_2(E - \mathrm{i}\,B) = 0. \tag{A.3}$$

Thus a complex-conjugate pair of equations arise. This result can be generalized to tensors of any type without loss of generality provided the invariance is satisfied. The matrix $M$ needs to be written in block form with blocks of zeros or the identity matrix to compensate for the number of dimensions.

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