1+1+2 electromagnetic perturbations on non-vacuum LRS class II spacetimes: decoupling scalar and 2-vector harmonic amplitudes

R B Burston

Max Planck Institute for Solar System Research, 37191 Katlenburg-Lindau, Germany

E-mail: burston@mps.mpg.de

Abstract

We use the covariant and gauge-invariant 1+1+2 formalism of Clarkson and
Barrett [1] to analyze electromagnetic (EM) perturbations on non-vacuum
locally rotationally symmetric (LRS) class II spacetimes. Ultimately, we
show how to derive six real decoupled equations governing the total of six
EM scalar and 2-vector harmonic amplitudes. Four of these are new, and
result from expanding the complex EM 2-vector which we defined in [2]
in terms of EM 2-vector harmonic amplitudes. We are then able to show
that there are four precise combinations of the amplitudes that decouple,
two of these are polar perturbations whereas the remaining two are axial.
The remaining two decoupled equations are the generalized Regge–Wheeler
equations which were developed previously in [3], and these govern the two
EM scalar harmonic amplitudes. However, our analysis generalizes this by
including a full description and classification of energy–momentum sources,
such as charges and currents.

PACS numbers: 04.25.Nx, 04.20.−q, 04.40.−b, 03.50.De, 04.20.Cv

1. Introduction

There has been recent interest [1–4] in the analysis of first-order perturbations to *locally
rotational symmetric* (LRS) class II spacetimes [5–7] using Clarkson and Barrett's 1+1+2
formalism [1]. Electromagnetic (EM) perturbations to non-vacuum LRS class II spacetimes
via the 1+1+2 formalism were first analyzed in [3]. Therein they derived covariant and gauge-
invariant generalized Regge–Wheeler (RW) [8] equations governing the radial parts of the
electric and magnetic fields, $\mathscr{E}$ and $\mathscr{B}$. They also presented a new scalar and vector harmonic
expansion formalism that naturally generalizes the *spherical* harmonic formalism developed
in [1]. Finally, they presented a detailed analysis of the EM scalar harmonic amplitudes $\mathscr{E}_{\mathrm{S}}$ and
$\mathscr{B}_{\mathrm{S}}$, *in the absence of energy–momentum sources*. Following this, we showed that expressing

Maxwell's equations in a 1+1+2 complex form is conducive to deriving a new decoupled, covariant and gauge-invariant complex equation governing a new complex EM 2-vector [2] $\Phi_\mu := \mathcal{E}_\mu + \mathrm{i}\,\mathcal{B}_\mu$. Thus clearly demonstrating that the EM 2-vector fields decouple from the scalar fields.

The notion of expanding first-order perturbation quantities in terms of scalar, vector and tensor harmonics has been used extensively throughout the literature. Thorne [9] provides an excellent review of the many different formalisms, conventions and notations used for spherical harmonics that were in existence up until 1980. There are more recent formalisms that generalize this concept, including Clarkson and Barrett's that is implemented in this paper, and this idea is based on those in [10]. Kodama *et al* [11, 12] also make use of scalar, vector and tensor harmonic expansions defined in [13–15] to analyze perturbations of higher dimensional spacetimes. In particular, they analyze EM and metric perturbations of charged black holes in n dimensions in [12] and show that only specific combinations of the first-order quantities satisfy decoupled equations, as was demonstrated for the 4-dimensional case by [16, 17].

In this paper, a vector harmonic expansion of the complex EM 2-vector is used and the governing equation yields a system of two equations which are coupled by the complex EM 2-vector harmonic amplitudes, Φ_V and $\bar{\Phi}_V$. We then discuss the invariance properties of this new coupled system and note that it is precisely analogous to the invariance properties of the 1+1+2 EM system we discussed in [2]. Therefore, precisely the same linear algebra techniques as in [2] are used to choose new complex variables that are successful for decoupling the system. It is then possible to separate the real from the imaginary parts and this ultimately results in four *real* decoupled equations governing four specific combinations of the *real* EM 2-vector harmonic amplitudes, $\mathcal{E}_V$, $\mathcal{B}_V$, $\bar{\mathcal{E}}_V$ and $\bar{\mathcal{B}}_V$, with a complete description of energy–momentum sources.

Furthermore, this analysis is supplemented by the generalized RW equations as presented in [3]. However, the equations are generalized here by explicitly writing the scalar harmonic expansion of the generalized RW equations and also include a treatment of the energy–momentum sources. Summarily, we find six *real* decoupled equations governing the six *real* scalar and 2-vector harmonic amplitudes, with a full scalar and 2-vector harmonic analysis of the energy–momentum source terms. Moreover, all quantities are neatly categorized into axial and polar perturbations.

Finally, we adhere to all notations and conventions as presented in [1, 3] and this differs from that employed in [2].

2. Previous work

2.1. Betschart and Clarkson's 1+1+2 non-vacuum LRS class II spacetime

The 1+1+2 scalars, and equations, governing background LRS class II spacetimes were initiated in [1] for the covariant Schwarzschild spacetime, and later generalized by Betschart and Clarkson in [3] for arbitrary non-vacuum LRS class II spacetimes. There is a set of ten non-vanishing scalars given by

$$\text{LRS class II:} \quad \{\mathcal{A}, \theta, \phi, \Sigma, \mathcal{E}, \mu, p, \mathcal{Q}, \Pi, \Lambda\}. \tag{1}$$

Here, $\mathcal{A}$ is the radial acceleration of the 4-velocity, θ and ϕ are respectively the expansions of the 3-sheets and 2-sheets, Σ is the radial part of the shear of the 3-sheet and $\mathcal{E}$ is the radial component of the gravito-electric tensor. The energy–momentum quantities, mass–energy density, pressure, radial heat flux and radial anisotropic stress are denoted respectively μ, p, $\mathcal{Q}$ and Π, and finally, Λ is the cosmological constant. These scalars do not vary over the

2-sheets and consequently, the covariant 2-derivative associated with the 2-sheets, which is denoted by δ_μ, of any of these scalars will vanish. Also, the equations governing these scalars arise from the Ricci identities and the Bianchi identities, and are reproduced here from [3]

$$\hat{\phi} + \tfrac{1}{2}\phi^2 + \left(\Sigma - \tfrac{2}{3}\theta\right)\left(\Sigma + \tfrac{1}{3}\theta\right) + \mathcal{E} = -\tfrac{2}{3}(\mu + \Lambda) - \tfrac{1}{2}\Pi, \tag{2}$$

$$\hat{\Sigma} - \tfrac{2}{3}\hat{\theta} + \tfrac{3}{2}\phi\Sigma = -\mathcal{Q}, \tag{3}$$

$$\hat{\mathcal{E}} + \tfrac{3}{2}\phi\mathcal{E} = \tfrac{1}{3}\hat{\mu} + \tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\mathcal{Q} - \tfrac{1}{2}\hat{\Pi} - \tfrac{3}{4}\phi\Pi, \tag{4}$$

$$\dot{\phi} + \left(\Sigma - \tfrac{2}{3}\theta\right)\left(\mathcal{A} - \tfrac{1}{2}\phi\right) = \mathcal{Q}, \tag{5}$$

$$\dot{\Sigma} - \tfrac{2}{3}\dot{\theta} - \tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)^2 + \mathcal{A}\phi + \mathcal{E} = \tfrac{1}{3}(\mu + 3p - 2\Lambda) + \tfrac{1}{2}\Pi, \tag{6}$$

$$\dot{\mathcal{E}} - \tfrac{3}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\mathcal{E} = \tfrac{1}{3}\dot{\mu} - \tfrac{1}{2}\dot{\Pi} + \tfrac{1}{4}\left(\Sigma - \tfrac{2}{3}\theta\right)\Pi + \tfrac{1}{2}\phi\mathcal{Q} - \tfrac{1}{2}(\mu + p)\left(\Sigma - \tfrac{2}{3}\theta\right), \tag{7}$$

$$\hat{\mathcal{A}} + (\mathcal{A} + \phi)\mathcal{A} - \dot{\theta} - \tfrac{1}{3}\theta^2 - \tfrac{3}{2}\Sigma^2 = \tfrac{1}{2}(\mu + 3p - 2\Lambda), \tag{8}$$

$$\dot{\mu} + \theta\mu + \hat{\mathcal{Q}} + (2\mathcal{A} + \phi)\mathcal{Q} + \theta p + \tfrac{3}{2}\Sigma\Pi = 0, \tag{9}$$

$$\dot{\mathcal{Q}} + \left(\Sigma + \tfrac{4}{3}\theta\right)\mathcal{Q} + \hat{p} + \mathcal{A}p + \hat{\Pi} + \left(\mathcal{A} + \tfrac{3}{2}\phi\right)\Pi + \mu\mathcal{A} = 0. \tag{10}$$

Here, the 'dot' derivative is defined $\dot{X}_{\mu\ldots\nu} := u^\alpha \nabla_\alpha X_{\mu\ldots\nu}$, where $X_{\mu\ldots\nu}$ represents any quantity. The 'hat' derivative is defined $\hat{W}_{\mu\ldots\nu} := n^\alpha D_\alpha W_{\mu\ldots\nu}$, where $W_{\mu\ldots\nu}$ represents a 3-tensor and D_μ is the covariant derivative associated with the 3-sheets.

Furthermore, the Gaussian curvature of the 2-sheet is also reproduced from [3] and is given by

$$K = \tfrac{1}{3}(\mu + \Lambda) - \mathcal{E} - \tfrac{1}{2}\Pi + \tfrac{1}{4}\phi^2 - \tfrac{1}{4}\left(\Sigma - \tfrac{2}{3}\theta\right)^2. \tag{11}$$

Thus, the background system of evolution, propagation and transportation equations are given by (2)–(10) and the background scalars (1) are assumed to become known quantities in the forthcoming first-order perturbation equations.

2.2. First-order Maxwell's equations in 1+1+2 complex form

We now consider first-order EM perturbations on LRS class II spacetimes given by the EM fields (E_μ and B_μ), charges (ρ_e) and the current (J_μ). Furthermore, these perturbations are regarded as gauge-invariant according to the Sachs–Stewart–Walker lemma [18, 19]. The first-order vectors are irreducibly split into 1+1+2 form according to

$$E_\mu = \mathcal{E}n_\mu + \mathcal{E}_\mu, \qquad B_\mu = \mathcal{B}n_\mu + \mathcal{B}_\mu \qquad \text{and} \qquad J_\mu = \mathcal{J}n_\mu + \mathcal{J}_\mu. \tag{12}$$

We showed in a recent paper [2] that Maxwell's first-order equations may be expressed in a new 1+1+2 complex form which is conducive to decoupling,

$$\hat{\Phi} + \phi\Phi + \delta^\alpha \Phi_\alpha = \rho_e, \tag{13}$$

$$\dot{\Phi} - \left(\Sigma - \tfrac{2}{3}\theta\right)\Phi + \mathrm{i}\,\epsilon^{\alpha\beta}\delta_\alpha \Phi_\beta = -\mathcal{J}, \tag{14}$$

$$\dot{\Phi}_{\bar{\mu}} + \left(\tfrac{1}{2}\Sigma + \tfrac{2}{3}\theta\right)\Phi_\mu - \mathrm{i}\epsilon_\mu{}^\alpha\left[\hat{\Phi}_\alpha + \left(\mathcal{A} + \tfrac{1}{2}\phi\right)\Phi_\alpha\right] + \mathrm{i}\epsilon_\mu{}^\alpha\delta_\alpha\Phi = -\mathcal{J}_\mu, \tag{15}$$

where i is the complex number and $\epsilon_{\mu\nu}$ is the Levi-Civita 2-tensor. Moreover, the complex EM scalar and the complex EM 2-vector have been defined,

$$\Phi := \mathcal{E} + \mathrm{i}\,\mathcal{B} \qquad \text{and} \qquad \Phi_\mu := \mathcal{E}_\mu + \mathrm{i}\,\mathcal{B}_\mu. \tag{16}$$

Subsequently, we showed in [2] that the complex scalar and 2-vector naturally decouple. Furthermore, the gauge-invariant and covariant equations arise from the 1+1+2 complex system (13)–(15),

$$\ddot{\Phi} - \left(\Sigma - \tfrac{5}{3}\theta\right)\dot{\Phi} - \hat{\hat{\Phi}} - (\mathcal{A} + 2\phi)\hat{\Phi} - V\Phi = \mathcal{S}, \tag{17}$$

$$\ddot{\Phi}_{\bar{\mu}} - \left(\Sigma - \tfrac{5}{3}\theta\right)\dot{\Phi}_{\bar{\mu}} - \hat{\hat{\Phi}}_{\bar{\mu}} - (\mathcal{A} + 2\phi)\hat{\Phi}_{\bar{\mu}} - V_{(1)}\Phi_{\mu}$$
$$- \mathrm{i}\epsilon_{\mu}{}^{\alpha}[(2\mathcal{A} - \phi)\dot{\Phi}_{\alpha} - 3\Sigma\hat{\Phi}_{\alpha} - V_{(2)}\Phi_{\alpha}] = \mathcal{S}_{\mu}, \tag{18}$$

where terms related to the background potentials were defined

$$V := \delta^2 + 2K - \mu + p + \Pi - 2\Lambda, \tag{19}$$

$$V_{(1)} := \delta^2 + \mathcal{E} + \tfrac{1}{4}\phi^2 - \mathcal{A}^2 + \phi\mathcal{A} + \tfrac{7}{4}\Sigma^2 - \tfrac{2}{9}\theta^2 + \tfrac{2}{3}\theta\Sigma - \tfrac{1}{3}\mu + p - \tfrac{4}{3}\Lambda, \tag{20}$$

$$V_{(2)} := -\dot{\mathcal{A}} - \tfrac{1}{3}\hat{\theta} - \tfrac{2}{3}\theta(\phi + 2\mathcal{A}) + \tfrac{1}{2}\Sigma(\phi + 4\mathcal{A}), \tag{21}$$

and the complex first-order energy–momentum sources are

$$\mathcal{S} := -\hat{\rho}_e - (\phi + \mathcal{A})\rho_e - \dot{\mathcal{J}} - \theta\,\mathcal{J} + \mathrm{i}\,\epsilon^{\alpha\beta}\delta_{\alpha}\,\mathcal{J}_{\beta}, \tag{22}$$

$$\mathcal{S}_{\mu} := -\dot{\mathcal{J}}_{\bar{\mu}} + \tfrac{3}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)\mathcal{J}_{\mu} - \delta_{\mu}\rho_e + \mathrm{i}\epsilon_{\mu}{}^{\alpha}\left(\delta_{\alpha}\,\mathcal{J} - \hat{\mathcal{J}}_{\alpha} - \tfrac{3}{2}\phi\,\mathcal{J}_{\alpha}\right), \tag{23}$$

where the 2-Laplacian is $\delta^2 := \delta^{\alpha}\delta_{\alpha}$. Thus, (17)–(18) clearly demonstrates the decoupling of the complex scalar and 2-vector. Moreover, (17) is the generalized RW equation in complex form as presented in [2]. Then a further decoupling between the EM scalars $\mathcal{E}$ and $\mathcal{B}$ can be achieved by taking the real and imaginary parts of (17) separately, which then correspond to those previously derived in [3].

2.3. Scalar and vector harmonics

The scalar and vector harmonic expansion for all 1+1+2 first-order quantities has been previously presented in [3] for non-vacuum LRS class II spacetimes and this was a natural generalization of the scalar and vector *spherical* harmonics given for the covariant Schwarzschild case in [1]. Here we state the necessary results from [3]. Dimensionless sheet harmonic functions Q (defined on the background) are reproduced here from [3]

$$\delta^2 Q = -\frac{k^2}{r^2}Q, \qquad \hat{Q} = \dot{Q} = 0, \tag{24}$$

where k^2 is real. The scalar, r, is covariantly defined by the following differential equations:

$$\hat{r} - \tfrac{1}{2}\phi r = 0, \qquad \dot{r} + \tfrac{1}{2}\left(\Sigma - \tfrac{2}{3}\theta\right)r = 0 \qquad \text{and} \qquad \delta_{\mu}r = 0. \tag{25}$$

Thus any first-order scalar function is expanded according to

$$\psi = \sum_{k}\psi_{\mathrm{S}}^{(k)}Q^{(k)} = \psi_{\mathrm{S}}Q, \tag{26}$$

where the subscript, S, indicates that a scalar harmonic expansion has been made. Furthermore, the summation over k is implicit in the last equality and ψ_{S} is referred to as the scalar harmonic amplitude, or just scalar amplitude.

Similarly, all vectors are expanded in terms of even (Q_{μ}) and odd ($\bar{Q}_{\mu}$) parity vector harmonics which are defined, respectively,

$$Q_{\mu} = r\delta_{\mu}Q \quad \rightarrow \quad \delta^2 Q_{\mu} = \left(K - \frac{k^2}{r^2}\right)Q_{\mu}, \tag{27}$$

$$\bar{Q}_{\mu} = r\epsilon_{\mu}{}^{\alpha}\delta_{\alpha}Q \quad \rightarrow \quad \delta^2 \bar{Q}_{\mu} = \left(K - \frac{k^2}{r^2}\right)\bar{Q}_{\mu}. \tag{28}$$

However, there is a subtle difference with the equations presented here from [3]. Note that 2-Laplacian operating on the vector harmonics in (27)–(28) are left in terms of the Gaussian curvature, whereas in [3] they use a further constraint of $K = \frac{1}{r^2}$ which is equivalent to choosing a particular normalization that was convenient for their analysis. Here we leave this normalization general.

Now, any first-order vector may be expanded according to

$$\psi_\mu = \sum_k \psi_V^{(k)} Q_\mu^{(k)} + \bar{\psi}_V^{(k)} \bar{Q}_\mu^{(k)} = \psi_V Q_\mu + \bar{\psi}_V \bar{Q}_\mu, \tag{29}$$

where the V is indicative of a 2-vector harmonic expansion. Also, the summation in the last quantity is implied, and ψ_V and $\bar{\psi}_V$ are the 2-vector harmonic amplitudes or 2-vector amplitudes.

3. Scalar harmonic expansion of the complex RW equation

A scalar (and vector) harmonic expansion of the EM scalars has been previously studied in [3] for vanishing energy–momentum sources. Here, our analysis generalizes those results by explicitly writing the generalized RW equation in a scalar harmonic form and furthermore, we include a treatment of energy–momentum sources. The complex EM scalar, Φ, and energy–momentum source, $\mathcal{S}$, are expanded in terms of scalar harmonics according to

$$\Phi = \Phi_S Q \qquad \text{and} \qquad \mathcal{S} = \mathcal{S}_S Q. \tag{30}$$

Similarly, the first-order charges and currents become

$$\rho_e = \rho_{eS} Q, \qquad \mathcal{J} = \mathcal{J}_S Q \qquad \text{and} \qquad \mathcal{J}_\mu = \mathcal{J}_V Q_\mu + \bar{\mathcal{J}}_V \bar{Q}_\mu. \tag{31}$$

Therefore, the complex generalized RW equation (17) becomes

$$\ddot{\Phi}_S - \left(\Sigma - \tfrac{5}{3}\theta\right)\dot{\Phi}_S - \hat{\hat{\Phi}}_S - (\mathcal{A} + 2\phi)\hat{\Phi}_S - V\Phi_S = \mathcal{S}_S, \tag{32}$$

where the potential and source are now

$$V = -\frac{k^2}{r^2} + 2K - \mu + p + \Pi - 2\Lambda, \tag{33}$$

$$\mathcal{S}_S = -\dot{\rho}_{eS} - (\phi + \mathcal{A})\rho_{eS} - \dot{\mathcal{J}}_S - \theta\,\mathcal{J}_S + i\,\frac{k^2}{r}\,\bar{\mathcal{J}}_V. \tag{34}$$

Therefore, since Φ_S and $\mathcal{S}_S$ are the *only* complex quantities in (32), the real and imaginary parts can be taken separately to obtain two real decoupled equations

$$\ddot{\mathcal{E}}_S - \left(\Sigma - \frac{5}{3}\theta\right)\dot{\mathcal{E}}_S - \hat{\hat{\mathcal{E}}}_S - (\mathcal{A} + 2\phi)\hat{\mathcal{E}}_S - V\mathcal{E}_S = -\dot{\rho}_{eS} - (\phi + \mathcal{A})\rho_{eS} - \dot{\mathcal{J}}_S - \theta\,\mathcal{J}_S, \tag{35}$$

$$\ddot{\mathcal{B}}_S - \left(\Sigma - \frac{5}{3}\theta\right)\dot{\mathcal{B}}_S - \hat{\hat{\mathcal{B}}}_S - (\mathcal{A} + 2\phi)\hat{\mathcal{B}}_S - V\mathcal{B}_S = \frac{k^2}{r}\,\bar{\mathcal{J}}_V. \tag{36}$$

Furthermore, we can see that the perturbations neatly separate into polar and axial perturbations which was similarly noted in [3] and are, respectively,

$$\text{Decoupled polar perturbation:} \quad \{\mathcal{E}_S\}, \tag{37}$$

$$\text{Decoupled axial perturbation:} \quad \{\mathcal{B}_S\}. \tag{38}$$

Here, we also further categorize the energy–momentum sources according to

$$\text{Energy–momentum polar perturbations:} \quad \{\rho_{e\text{S}}, \mathscr{J}_\text{S}\}, \tag{39}$$

$$\text{Energy–momentum axial perturbations:} \quad \{\bar{\mathscr{J}}_\text{V}\}. \tag{40}$$

At this stage it is also clear that the remaining energy–momentum quantity ($\mathscr{J}_\text{V}$) must be a polar perturbation since its partner ($\bar{\mathscr{J}}_\text{V}$) is axial. We will see that this is consistent with the vector harmonic expansions in the following section.

As a final note, we inspect the energy–momentum source terms in (35) and (36) and note that the polar electric scalar amplitude, $\mathscr{E}_\text{S}$, is being forced purely by the polar energy–momentum scalar amplitudes, $\rho_{e\text{S}}$ and $\mathscr{J}_\text{S}$. Furthermore, the axial magnetic scalar amplitude, $\mathscr{B}_\text{S}$, is being forced purely by the axial current 2-vector amplitude, $\bar{\mathscr{J}}_\text{V}$, and completely vanishes in the cases where the harmonic summation index, k^2, also vanishes.

4. Vector harmonic expansion

Much more work is required to decouple the EM 2-vector amplitudes, $\mathscr{E}_\text{V}$ and $\mathscr{B}_\text{V}$. Firstly, the complex EM 2-tensor (16) and the energy–momentum source (23) are expanded according to

$$\Phi_\mu = \Phi_\text{V} Q_\mu + \bar{\Phi}_\text{V} \bar{Q}_\mu \qquad \text{and} \qquad \mathcal{S}_\mu = \mathcal{S}_\text{V} Q_\mu + \bar{\mathcal{S}}_\text{V} \bar{Q}_\mu. \tag{41}$$

Thus, by substituting (41) into (18) we get a coupled system of the form

$$\ddot{\Phi}_\text{V} - \left(\Sigma - \tfrac{5}{3}\theta\right)\dot{\Phi}_\text{V} - \hat{\hat{\Phi}}_\text{V} - (\mathcal{A} + 2\phi)\hat{\Phi}_\text{V} - V_{(1)}\Phi_\text{V}$$
$$+ \text{i}\,[(2\mathcal{A} - \phi)\dot{\bar{\Phi}}_\text{V} - 3\Sigma\hat{\bar{\Phi}}_\text{V} - V_{(2)}\bar{\Phi}_\text{V}] = \mathcal{S}_\text{V}, \tag{42}$$

$$\ddot{\bar{\Phi}}_\text{V} - \left(\Sigma - \tfrac{5}{3}\theta\right)\dot{\bar{\Phi}}_\text{V} - \hat{\hat{\bar{\Phi}}}_\text{V} - (\mathcal{A} + 2\phi)\hat{\bar{\Phi}}_\text{V} - V_{(1)}\bar{\Phi}_\text{V}$$
$$- \text{i}[(2\mathcal{A} - \phi)\dot{\Phi}_\text{V} - 3\Sigma\hat{\Phi}_\text{V} - V_{(2)}\Phi_\text{V}] = \bar{\mathcal{S}}_\text{V}. \tag{43}$$

The background potential and energy–momentum sources similarly become

$$V_{(1)} := -\frac{k^2}{r^2} + K + \mathcal{E} + \frac{1}{4}\phi^2 - \mathcal{A}^2 + \phi\mathcal{A} + \frac{7}{4}\Sigma^2 - \frac{2}{9}\theta^2 + \frac{2}{3}\theta\Sigma - \frac{1}{3}\mu + p - \frac{4}{3}\Lambda, \tag{44}$$

$$\mathcal{S}_\text{V} := -\dot{\mathscr{J}}_\text{V} + \frac{3}{2}\left(\Sigma - \frac{2}{3}\theta\right)\mathscr{J}_\text{V} - \frac{1}{r}\rho_{e\text{S}} + \text{i}\left(\hat{\bar{\mathscr{J}}}_\text{V} + \frac{3}{2}\phi\,\bar{\mathscr{J}}_\text{V}\right), \tag{45}$$

$$\bar{\mathcal{S}}_\text{V} := -\dot{\bar{\mathscr{J}}}_\text{V} + \frac{3}{2}\left(\Sigma - \frac{2}{3}\theta\right)\bar{\mathscr{J}}_\text{V} + \text{i}\left(\frac{1}{r}\mathscr{J}_\text{S} - \hat{\mathscr{J}}_\text{V} - \frac{3}{2}\phi\,\mathscr{J}_\text{V}\right). \tag{46}$$

By inspection, equations (42)–(43) are (momentarily ignoring sources) now precisely invariant under the simultaneous transformation of the form $\Phi_\text{V} \rightarrow \bar{\Phi}_\text{V}$ and $\bar{\Phi}_\text{V} \rightarrow -\Phi_\text{V}$. Thus, this is precisely the same invariance exhibited by the 1+1+2 EM system as discussed in [2] and decoupling may be achieved by choosing new independent variables and sources according to

$$\Phi_- := \Phi_\text{V} - \text{i}\,\bar{\Phi}_\text{V} \qquad \text{and} \qquad \Phi_+ := \Phi_\text{V} + \text{i}\,\bar{\Phi}_\text{V}, \tag{47}$$

and similarly for the source, $\mathcal{S}_\pm := \mathcal{S}_\text{V} \pm \text{i}\,\bar{\mathcal{S}}_\text{V}$, where '$\pm$' is relative. Consequently, by taking complex additions and subtractions of (42)–(43) we find two decoupled complex equations, one for each of Φ_+ and Φ_-,

$$\ddot{\Phi}_+ - \left[\Sigma - \tfrac{5}{3}\theta - (2\mathcal{A} - \phi)\right]\dot{\Phi}_+ - \hat{\hat{\Phi}}_+ - (\mathcal{A} + 2\phi + 3\Sigma)\hat{\Phi}_+ - [V_{(1)} + V_{(2)}]\Phi_+ = \mathcal{S}_+, \tag{48}$$

$$\ddot{\Phi}_- - \left[\Sigma - \tfrac{5}{3}\theta + (2\mathcal{A} - \phi)\right]\dot{\Phi}_- - \hat{\hat{\Phi}}_- - (\mathcal{A} + 2\phi - 3\Sigma)\hat{\Phi}_- - [V_{(1)} - V_{(2)}]\Phi_- = \mathcal{S}_-. \quad (49)$$

Therefore, since the only complex quantities in (48)–(49) are Φ_+, Φ_-, $\mathcal{S}_+$ and $\mathcal{S}_-$, the real and imaginary parts may be considered separately and there are actually four *real* decoupled equations, one for each of the real and imaginary parts of $\Phi_\pm$, i.e. $\Re[\Phi_+]$, $\Im[\Phi_+]$, $\Re[\Phi_-]$ and $\Im[\Phi_-]$. These quantities can now be related back to the harmonic amplitudes of the real EM 2-vector. By using (47), and noting that according to (16)

$$\Phi_V := \mathcal{E}_V + i\mathcal{B} \qquad \text{and} \qquad \bar{\Phi}_V := \bar{\mathcal{E}}_V + i\bar{\mathcal{B}}, \quad (50)$$

then the real and imaginary parts of $\Phi_\pm$ are

$$\Phi_+ = (\mathcal{E}_V - \bar{\mathcal{B}}_V) + i(\mathcal{B}_V + \bar{\mathcal{E}}_V) \qquad \text{and} \qquad \Phi_- = (\mathcal{B}_V + \bar{\mathcal{E}}_V) + i(\mathcal{B}_V - \bar{\mathcal{E}}_V), \quad (51)$$

and the sources

$$\mathcal{S}_+ := -\frac{1}{r}(\rho_{e\,S} + \mathscr{J}_S) - \dot{\mathscr{J}}_V + \hat{\mathscr{J}}_V + \frac{3}{2}\left(\Sigma - \frac{2}{3}\theta + \phi\right)\mathscr{J}_V$$
$$- i\left[\dot{\bar{\mathscr{J}}}_V - \hat{\bar{\mathscr{J}}}_V - \frac{3}{2}\left(\Sigma - \frac{2}{3}\theta + \phi\right)\bar{\mathscr{J}}_V\right], \quad (52)$$

$$\mathcal{S}_- := -\frac{1}{r}(\rho_{e\,S} - \mathscr{J}_S) - \dot{\mathscr{J}}_V - \hat{\mathscr{J}}_V + \frac{3}{2}\left(\Sigma - \frac{2}{3}\theta - \phi\right)\mathscr{J}_V$$
$$+ i\left[\dot{\bar{\mathscr{J}}}_V + \hat{\bar{\mathscr{J}}}_V - \frac{3}{2}\left(\Sigma - \frac{2}{3}\theta - \phi\right)\bar{\mathscr{J}}_V\right]. \quad (53)$$

Therefore by inspecting (51), it is now clear that we have four different combinations of the EM 2-vector amplitudes which decouple. Furthermore, they may also be categorized into polar and axial perturbations. Summarily, the precise combinations of the harmonic amplitudes of the real EM 2-vectors that individually decouple are

$$\text{Decoupled polar perturbations:} \quad \{\mathcal{E}_V + \bar{\mathcal{B}}_V, \mathcal{E}_V - \bar{\mathcal{B}}_V\}, \quad (54)$$

$$\text{Decoupled axial perturbations:} \quad \{\mathcal{B}_V + \bar{\mathcal{E}}_V, \mathcal{B}_V - \bar{\mathcal{E}}_V\}. \quad (55)$$

Furthermore, this also elucidates that the remaining energy–momentum quantity yet to be categorized is a polar perturbation as anticipated,

$$\text{Energy–momentum polar perturbation:} \quad \{\mathscr{J}_V\}. \quad (56)$$

Moreover, by inspecting (54)–(55), it is clear that if one were to integrate the four decoupled equations (48)–(49), it is then possible to construct linear combinations of the solutions to ultimately find each of $\mathcal{E}_V$, $\mathcal{B}_V$, $\bar{\mathcal{E}}_V$ and $\bar{\mathcal{B}}_V$.

5. Summary

We have provided a complete analysis of the EM perturbations to the most general non-vacuum LRS class II spacetimes, using the complex 1+1+2 formalism and a harmonic scalar and vector expansion of the complex EM scalar and 2-vector. We derived four new decoupled equations which govern four precise combinations of the EM 2-vector amplitudes and this has a full energy–momentum treatment. Furthermore, we generalized the two RW equations, which govern the scalar amplitudes, from [3] to also include energy–momentum quantities. This gives a total of six decoupled equations governing the combined total of six EM amplitudes. Summarily, the necessary combinations that individually decouple are

$$\text{Decoupled polar perturbations:} \quad \{\mathscr{E}_\mathrm{S}, \mathscr{E}_\mathrm{V} + \bar{\mathscr{B}}_\mathrm{V}, \mathscr{E}_\mathrm{V} - \bar{\mathscr{B}}_\mathrm{V}\}, \tag{57}$$

$$\text{Decoupled axial perturbations:} \quad \{\mathscr{B}_\mathrm{S}, \mathscr{B}_\mathrm{V} + \bar{\mathscr{E}}_\mathrm{V}, \mathscr{B}_\mathrm{V} - \bar{\mathscr{E}}_\mathrm{V}\}. \tag{58}$$

Furthermore, the energy–momentum sources also fall into polar and axial perturbations according to

$$\text{Energy–momentum polar perturbations:} \quad \{\rho_{e\mathrm{S}}, \mathscr{J}_\mathrm{s}, \mathscr{J}_\mathrm{v}\}, \tag{59}$$

$$\text{Energy–momentum axial perturbations:} \quad \{\bar{\mathscr{J}}_\mathrm{V}\}. \tag{60}$$

We present the main results, i.e. the decoupled equations (35)–(36) and (48)–(49), as a single master equation of the form

$$\ddot{\Psi} - \left[\Sigma - \tfrac{5}{3}\theta - \sigma(2\mathcal{A} - \phi)\right]\dot{\Psi} - \hat{\hat{\Psi}} - (\mathcal{A} + 2\phi + \sigma 3\Sigma)\hat{\Psi} - U\Psi = \mathscr{S}, \tag{61}$$

where for

polar perturbations:

$$\Psi = \mathscr{E}_\mathrm{S} \quad \text{for} \quad \sigma = 0, \quad U = V \quad \text{and} \quad \mathscr{S} = \Re[\mathcal{S}_\mathrm{S}], \tag{62}$$

$$\Psi = (\mathscr{E}_\mathrm{V} - \bar{\mathscr{B}}_\mathrm{V}) \quad \text{for} \quad \sigma = 1, \quad U = V_{(1)} + V_{(2)} \quad \text{and} \quad \mathscr{S} = \Re[\mathcal{S}_+], \tag{63}$$

$$\Psi = (\mathscr{E}_\mathrm{V} + \bar{\mathscr{B}}_\mathrm{V}) \quad \text{for} \quad \sigma = 1, \quad U = V_{(1)} + V_{(2)} \quad \text{and} \quad \mathscr{S} = \Re[\mathcal{S}_-], \tag{64}$$

axial perturbations:

$$\Psi = \mathscr{B}_\mathrm{S} \quad \text{for} \quad \sigma = 0, \quad U = V \quad \text{and} \quad \mathscr{S} = \Im[\mathcal{S}_\mathrm{S}], \tag{65}$$

$$\Psi = (\bar{\mathscr{B}}_\mathrm{V} + \mathscr{E}_\mathrm{V}) \quad \text{for} \quad \sigma = -1, \quad U = V_{(1)} - V_{(2)} \quad \text{and} \quad \mathscr{S} = \Im[\mathcal{S}_+], \tag{66}$$

$$\Psi = (\mathscr{B}_\mathrm{V} - \bar{\mathscr{E}}_\mathrm{V}) \quad \text{for} \quad \sigma = -1, \quad U = V_{(1)} - V_{(2)} \quad \text{and} \quad \mathscr{S} = \Im[\mathcal{S}_-]. \tag{67}$$

It is anticipated that the decoupling analysis provided here will be useful for modeling spacetimes for which there are small EM fields, charges and currents present. Moreover, we also note that this analysis is the primary motivation to decouple the significantly harder problem of *gravitational* perturbations on LRS class II spacetimes using the *gravitoelectromagnetic* (GEM) formalism [20]. It is very well established that the GEM equations governing gravitational fields are remarkably similar to the EM equations governing EM fields [21]. In a later paper, we will show that the EM 2-vector harmonic amplitudes behave in a very similar manner to the GEM 2-tensor amplitudes, and consequently, we decouple four precise combinations of the GEM 2-tensor harmonic amplitudes for gravitational perturbations.

References

[1] Clarkson C and Barrett R 2003 *Class. Quantum Grav.* **20** 3855–84
[2] Burston R B and Lun A W C 2006 *Preprint* gr-qc/0610062
[3] Betschart G and Clarkson C 2004 *Class. Quantum Grav.* **21** 5587–607
[4] Clarkson C *et al* 2004 *Astrophys. J.* **613** 492–505
[5] Ellis G F R 1967 *J. Math. Phys.* **8** 1171
[6] Stewart J M and Ellis G F R 1968 *J. Math. Phys.* **9** 1072
[7] Elst H and Ellis G F R 1996 *Class. Quantum Grav.* **13** 1099–127
[8] Regge T and Wheeler J 1957 *Phys. Rev.* **108** 1063
[9] Thorne K 1980 *Rev. Mod. Phys.* **52** 299–339

[10] Ellis G F R and van Elst H 1999 *Cosmological Models (Cargèse Lectures 1998)* ed M Lachièze-Rey (Dordrecht:
         Kluwer)
[11] Kodama H and Ishibashi A 2003 *Prog. Theor. Phys.* **110** 701–22
[12] Kodama H and Ishibashi A 2004 *Prog. Theor. Phys.* **111** 29–73
[13] Kodama H and Sasaki M 1984 *Prog. Theor. Phys. Suppl.* **78** 1–166
[14] Mukohyama S 2000 *Phys. Rev.* D **62** 084015
[15] Kodama H, Ishibashi A and Seto O 2000 *Phys. Rev.* D **62** 064022
[16] Zerilli F 1974 *Phs. Rev.* D **9** 860–68
[17] Moncrief V 1974 *Phys. Rev.* D **9** 2707–9
[18] Sachs R 1964 *Relativity, Groups and Topology* ed B DeWitt and C DeWitt (New York: Gordon and Breach)
[19] Stewart J M and Walker M 1974 *Proc. R. Soc.* **341** 49–74
[20] Bel L 1958 *C. R. Acad. Sci.* **247** 1094
[21] Maartens R and Bassett 1998 *Class. Quantum Grav.* **15** 705–17