Numerical Integration of Partial Differential Equations (PDEs)

- Introduction to PDEs.
- Semi-analytic methods to solve PDEs.
- Introduction to Finite Differences.
- Stationary Problems, Elliptic PDEs.
- Time dependent Problems.

- Complex Problems in Solar System Research.
Complex Problems in Solar System Research.

• Stationary Problems: Magneto-hydrostatic equilibria to model magnetic field and plasma in the solar corona.

• Time-dependent Problems: Multi-fluid-Maxwell simulation of plasmas (courtesy Nina Elkina)
Modeling the solar corona

• Magnetic fields structure the solar corona.
• But we cannot measure them directly.
• Solution: Solve PDEs and use photospheric magnetic field measurements to prescribe boundary conditions.

• Let’s start with the simplest approach:
  Potential fields: \( \nabla \times \mathbf{B} = 0, \nabla \cdot \mathbf{B} = 0 \)

With \( \mathbf{B} = \nabla f \) we have to solve a Laplace equation:

\[ \Delta f = 0 \]
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 f}{\partial \phi^2} = 0
\]

We try to solve this equation by separation of variables:

\[f(r, \theta, \phi) = f_1(r) \cdot f_2(\theta, \phi)\]

and after multiplication with \(\frac{r^2}{f_1(r)f_2(\theta, \phi)}\) we get:

\[
\frac{\partial}{\partial r} \left( r^2 \frac{\partial f_1(r)}{\partial r} \right) = l(l + 1) f_1(r)
\]

\[
\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial f_2}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 f_2}{\partial \phi^2} = -l(l + 1) f_2(\theta, \phi)
\]

The solutions of the radial part are:

\[f_1(r) = r^{-(l+1)}, \text{ and } f_1(r) = r^l\]

We can further separate the angular equation (and get another separation constant \(m\)) or just look in a text-book or Wikipedia and find that this equation is solved by spherical harmonics \(Y_{lm}(\theta, \phi)\)
The 3D-solution of the Laplace equation can be found by superposition of the particular solutions $f(r, \theta, \phi) = f_1(r) \cdot f_2(\theta, \phi)$ as:

$$f(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ A_{lm} \ r^l + B_{lm} \ r^{-(l+1)} \right] Y_{lm}(\theta, \phi)$$

where $Y_{lm}$ are Spherical Harmonics and $A_{lm}$ and $B_{lm}$ are coefficient which we prescribe from boundary conditions.

In the photosphere ($r = 1R_s$) the radial magnetic field $B_r(r = 0)$ is measured and used to prescribe von Neumann B.C. We make a spherical harmonic decomposition:

$$B_r(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} Y_{lm}(\theta, \phi)$$

$$C_{lm} = \int_{0}^{2\pi} \int_{0}^{\pi} Y_{lm}^* (\theta, \phi) \ B_r(\theta, \phi) \ \sin(\theta) \ d\theta d\phi$$

where $Y_{lm}^* = (-1)^m \ Y_{l,-m}$. 
Outer radial boundary at source surface \((r_1 \approx 2.5 R_s)\). We assume that the field becomes radial here: \(\vec{B} = B_r \vec{e}_r\) for \(r = r_1\):

\[
B_\theta = \frac{1}{r} \frac{\partial f(r, \theta, \phi)}{\partial \theta}
\]

\[
B_\phi = \frac{1}{r \sin(\theta)} \frac{\partial f(r, \theta, \phi)}{\partial \phi}
\]

are supposed to vanish at \(r = r_1\).

Together with the photospheric boundary condition we get two equation to calculate \(A_{lm}\) and \(B_{lm}\):

\[
A_{lm} l r_0^{(l-1)} - B_{lm} (l + 1) r_0^{-(l+2)} = C_{lm}
\]

\[
A_{lm} r_1^l + B_{lm} r_1^{-(l+1)} = 0
\]

which lead to:

\[
A_{lm} = \frac{C_{lm} r_0^{2+l}}{r_1^{1+2l} + l \left( r_0^{1+2l} + r_1^{1+2l} \right)}
\]

\[
B_{lm} = - \left( \frac{C_{lm} r_0^{2+l} r_1^{1+2l}}{r_1^{1+2l} + l \left( r_0^{1+2l} + r_1^{1+2l} \right)} \right)
\]
Solution of Laplace equation for potential coronal magnetic fields:

\[ f(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ A_{lm} \, r^l + B_{lm} \, r^{-(l+1)} \right] Y_{lm}(\theta, \phi) \]

\[ B_r = \frac{\partial f}{\partial r} \]
\[ B_\theta = \frac{1}{r} \frac{\partial f}{\partial \theta} \]
\[ B_\phi = \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \phi} \]

Show example in IDL
Nonlinear Force-Free Fields

• Potential fields give impression about global topology of the coronal magnetic field.
• But: Approach is too simple to describe magnetic field and energy in active regions accurately.
• We include field aligned electric currents, the (nonlinear) force-free approach.

\[
(\nabla \times \mathbf{B}) \times \mathbf{B} = 0
\]
\[
\nabla \cdot \mathbf{B} = 0
\]
\[
\nabla \times \mathbf{B} = \alpha \mathbf{B}
\]
\[
\mathbf{B} \cdot \nabla \alpha = 0
\]
Nonlinear Force-Free Fields
(direct upward integration)

\[ \nabla \times \mathbf{B} = \alpha \mathbf{B} \]
\[ \mathbf{B} \cdot \nabla \alpha = 0 \]

Wu et al. 1990 proposed to solve these equations by upward integration:

- Compute vertical current in photosphere \((z=0)\)
- Compute alpha
- Compute horizontal currents
- Integrate \(\mathbf{B}\) upwards
- Repeat all steps for \(z=1,2,...\)

\[
\mu_0 j_{z0} = \frac{\partial B_{y0}}{\partial x} - \frac{\partial B_{x0}}{\partial y}
\]

\[
\alpha_0 = \frac{j_{z0}}{B_{z0}}
\]

\[
j_{x0} = \alpha_0 B_{x0}, \quad j_{y0} = \alpha_0 B_{y0}
\]

\[
\frac{\partial B_{x0}}{\partial z} = j_{y0} + \frac{\partial B_{z0}}{\partial x},
\]

\[
\frac{\partial B_{y0}}{\partial z} = \frac{\partial B_{z0}}{\partial y} - j_{x0},
\]

\[
\frac{\partial B_{z0}}{\partial z} = -\frac{\partial B_{x0}}{\partial x} - \frac{\partial B_{y0}}{\partial y}.
\]
Nonlinear Force-Free Fields
(direct upward integration)

- Straight forward scheme.
- Easy to implement.
- But: Not useful because the method is unstable.
- Why?
- Ill-posed problem.
Why is the problem ill-posed?

- Problem-1: Measured Magnetic field in photosphere is not force-free consistent.
  - Cure: We do regularization (or preprocessing) to prescribe consistent boundary conditions.
- Problem-2: Even for ‘ideal consistent’ data the upward integration is unstable (exponential growing modes blow up solution).
  - Cure: Reformulate the equations and apply a stable (iterative) method.
Consistency criteria for boundary-data (Aly 1989)

If these relations are NOT fulfilled, then the boundary data are inconsistent with the nonlinear force-free PDEs.

Ill posed Problem.
Preprocessing or Regularization

(Wiegelmann et al. 2006)

**Input:** Measured ill posed data => **Output:** Consistent B.C.

\[ L_{cp} = \mu_1 L_1 + \mu_2 L_2 + \mu_3 L_3 + \mu_4 L_4 \]

\[ L_1 = \left( \sum_p B_x B_z \right)^2 + \left( \sum_p B_y B_z \right)^2 + \left( \sum_p B_z^2 - B_x^2 - B_y^2 \right)^2 \]

\[ L_2 = \left( \sum_p x(B_z^2 - B_x^2 - B_y^2) \right)^2 + \left( \sum_p y(B_z^2 - B_x^2 - B_y^2) \right)^2 + \left( \sum_p (yB_x B_z - xB_y B_z) \right)^2 \]

\[ L_3 = \sum_p (B_x - B_{xobs})^2 + \sum_p (B_y - B_{yobs})^2 + \sum_p (B_z - B_{zobs})^2 \]

\[ L_4 = \sum_p (\Delta B_x)^2 + \sum_p (\Delta B_y)^2 + \sum_p (\Delta B_z)^2 \]
Non-linear Force-Free Fields

Force-free magnetic fields have to obey

\[(\nabla \times B) \times B = 0, \ \nabla \cdot B = 0\]

We define the functional (Wheatland, Sturrock, Roumeliotis 2000)

\[L = \int_V w(x, y, z) \left[ B^{-2} |(\nabla \times B) \times B|^2 + |\nabla \cdot B|^2 \right] d^3x\]

\(w\) is a weighting function (Wiegelmann 2004).

We minimize \(L\):

\[\frac{1}{2} \frac{dL}{dt} = -\int_V \frac{\partial B}{\partial t} \cdot \tilde{F} \ d^3x - \int_s \frac{\partial B}{\partial t} \cdot \tilde{G} \ d^2x\]

If all components of \(B\) are fixed on the boundaries of a computational box we get an evolution equation for \(B\)

\[\frac{\partial B}{\partial t} = \mu \tilde{F}\]
\[ \tilde{F} = w \ F + (\Omega_a \times B) \times \nabla w + (\Omega_b \cdot B) \ \nabla w \]
\[ \tilde{G} = w \ G \]

\[ F = \nabla \times (\Omega_a \times B) - \Omega_a \times (\nabla \times B) \]
\[ + \nabla (\Omega_b \cdot B) - \Omega_b (\nabla \cdot B) + (\Omega_a^2 + \Omega_b^2) B \]

\[ G = \hat{n} \times (\Omega_a \times B) - \hat{n} (\Omega_b \cdot B), \]
\[ \Omega_a = B^{-2} \ [(\nabla \times B) \times B] \]
\[ \Omega_b = B^{-2} \ [(\nabla \cdot B) B]. \]
Preprocessing

Consistent Boundaries

Solve (Force-Free) PDEs

Coronal Magnetic Field

Measure data
Flaring Active Region
(Thalmann & Wiegelmann 2008)

M6.1 Flare

Quiet Active Region

Solar X-ray flux. Vertical blue lines: vector magnetograms available

Magnetic field extrapolations from Solar Flare telescope

Extrapolated from SOLIS vector magnetograph
Magnetohydrostatics

Model magnetic field and plasma consistently:

\[
(\nabla \times \mathbf{B}) \times \mathbf{B} - \mu_0 \nabla p - \mu_0 \rho \nabla \Psi = 0
\]

- Lorentz force
- pressure gradient
- gravity

\[
\nabla \cdot \mathbf{B} = 0
\]
We define the functional

\[ L(B, p, \rho) = \int \left[ \frac{|(\nabla \times B) \times B - \mu_0 \nabla p - \mu_0 \rho \nabla \Psi|^2}{B^2} + |\nabla \cdot B|^2 \right] r^2 \sin \theta \, dr \, d\theta \, d\phi \]

The magnetohydrostatic equations are fulfilled if $L=0$

For easier mathematical handling we use

\[ \Omega_a = B^{-2} \left[ (\nabla \times B) \times B - \mu_0 \nabla p - \mu_0 \rho \nabla \Psi \right] \]
\[ \Omega_b = B^{-2} \left[ (\nabla \cdot B) B \right], \]

and rewrite $L$ as

\[ L = \int_V B^2 \Omega_a^2 + B^2 \Omega_b^2 \, r^2 \sin \theta \, dr \, d\theta \, d\phi. \]
Taking the derivative of $L$ with respect to an iteration parameter $t$, where $\mathbf{B}$, $p$, $\rho$ are assumed to depend upon $t$, we obtain

$$
\frac{1}{2} \frac{dL}{dt} = - \int_V \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{F} \, dV + \int_V \frac{\partial p}{\partial t} \mu_0 \nabla \cdot \Omega_a \, dV
$$

$$
- \int_V \frac{\partial \rho}{\partial t} \mu_0 \Omega_a \cdot \nabla \Psi \, dV - \int_S \frac{\partial \mathbf{B}}{\partial t} \mathbf{G} \, dS
$$

$$
- \int_S \frac{\partial \rho}{\partial t} \mu_0 \Omega_a \cdot dS,
$$

Iterative Equations ensure monotonously decreasing functional $L$ for vanishing surface integrals (boundary conditions).
$$L(B, p, \rho) = \int \left[ \frac{|(\nabla \times B) \times B - \mu_0 \nabla p - \mu_0 \rho \nabla \Psi|^2}{B^2} + |\nabla \cdot B|^2 \right] \rho^2 \sin \theta \, dr \, d\theta \, d\phi$$
Modeling the solar corona
Summary

• First one has to **find appropriate PDEs** which are adequate to model (certain aspects of) the solar corona. Here: Stationary magnetic fields and plasma.

• Use measurements to prescribe boundary conditions.

• **Regularize** (preprocess) **data** to derive **consistent boundary conditions** for the chosen PDE.

• **Stationary equilibria** (solution of our PDEs) can be used as initial condition for time dependent computation of other PDEs (MHD-simulations, planned).
Multi-fluid-Maxwell simulation of plasmas (courtesy Nina Elkina)

- The kinetic Vlasov-Maxwell system.
- From 6D-Vlasov equation to 3D-fluid approach.
- Generalization of flux-conservative form.
- Lax-Wendroff + Slope limiter
- Application: Weibel instability
Kinetic approach for collisionless plasma

Vlasov equation for plasma species

$$\frac{df}{dt} = \frac{\partial f_\alpha}{\partial t} + \vec{v} \frac{\partial f_\alpha}{\partial \vec{r}} + \frac{q_\alpha}{m_\alpha} \left[ \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \right] \frac{\partial f_\alpha}{\partial \vec{v}} = 0$$

Maxwell equations for EM fields

$$\frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{B} - \frac{4\pi}{c} \sum_\alpha q_\alpha \int \vec{v} f_\alpha d\vec{v}$$

$$\frac{1}{c} \frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\nabla \cdot \vec{E} = 4\pi \sum_\alpha q_\alpha \int f_\alpha d\vec{v}$$

$$\nabla \cdot \vec{B} = 0$$

$$f(x, y, z, v_x, v_y, v_z, t)$$

3D + 3V = 6 dimensions + time
How to lose information?

Instead of all the details of the distribution of particles consider only a small number of velocity moments:

Density:

\[ n(x, t) = \int dv \ F = \sum_{i=1,N} \delta(x - x_i) \]

Momentum density:

\[ n(x, t)u(x, t) = \int dv \ v \ F = \sum_{i=1,N} v_i \delta(x - x_i) \]

Kinetic energy density:

\[ K(x, t) = \int dv \ \frac{m}{2} v^2 \ F = \sum_{i=1,N} \frac{m}{2} v_i^2 \delta(x - x_i) \]

Kinetic energy flux:

\[ Q(x, t) = \int dv \ \frac{m}{2} v^3 \ F = \sum_{i=1,N} \frac{m}{2} v_i^3 \delta(x - x_i) \]

etc…
The multifluid simulation code

\[
\begin{pmatrix}
    \rho \\
    \rho v_x \\
    \rho v_y \\
    \rho v_z \\
    \rho v_x v_x + P_{xx} \\
    \rho v_x v_y + P_{xy} \\
    \rho v_z v_x + P_{xz} \\
    \rho v_z v_y + P_{yz} \\
    \rho v_z v_z + P_{zz}
\end{pmatrix}
\begin{pmatrix}
    \frac{\partial}{\partial t} \\
    \frac{\partial}{\partial x} \\
    \frac{\partial}{\partial y} \\
    \frac{\partial}{\partial z} \\
    \frac{\partial}{\partial x} \\
    \frac{\partial}{\partial y} \\
    \frac{\partial}{\partial z} \\
    \frac{\partial}{\partial x} \\
    \frac{\partial}{\partial y}
\end{pmatrix}
\begin{pmatrix}
    \rho v_x \\
    \rho v_x v_x + P_{xx} \\
    \rho v_x v_y + P_{xy} \\
    \rho v_x v_z + P_{xz} \\
    \rho v_x v_x v_x + 3v_x P_{xx} \\
    \rho v_x v_x v_y + 2v_x P_{xy} + v_y P_{xx} \\
    \rho v_x v_x v_z + 2v_x P_{xz} + v_z P_{xx} \\
    \rho v_x v_y v_y + v_x P_{yy} + 2v_y P_{xy} \\
    \rho v_x v_y v_z + v_x P_{yz} + v_y P_{zz} + v_z P_{xy}
\end{pmatrix}
= \frac{q}{m}
\begin{pmatrix}
    0 \\
    \rho (E_z + v_y B_x - v_z B_y) \\
    \rho (E_y + v_z B_x - v_x B_z) \\
    \rho (E_x + v_z B_y - v_y B_z) \\
    2\rho v_x E_x + 2(B_z P_{yx} - B_y P_{xz}) \\
    \rho (v_x E_y + v_y E_x + (B_z P_{yx} - B_y P_{xz} + B_x P_{zz} + B_x P_{zz}) \\
    \rho (v_x E_y + v_y E_x + (B_z P_{yx} + B_y P_{xz} - B_y P_{zz} - B_x P_{xy}) \\
    2\rho v_z E_y + 2(B_x P_{yz} - B_x P_{yz}) \\
    \rho (v_y E_z + v_z E_y + (B_y P_{yx} - B_z P_{xz} + B_x P_{zz} - B_x P_{yz}) \\
    2\rho v_z E_z + 2(B_y P_{xz} - B_z P_{yz})
\end{pmatrix}
\]

...are solved with using high-resolution semi-discrete method.
These equations include also finite Larmor radii effect, pressure anisotropy, electron inertia, charge separation
Formally the multi-fluid-equations can be written in vector form

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S
\]

Fluxes

Source-Term

Generalized form of our flux-conservative equation:

\[
\frac{\partial u}{\partial t} = -\frac{\partial F(u)}{\partial x}
\]
\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S
\]

The individual terms are somewhat more complex as in our example advection equation.

\[
U = \begin{pmatrix}
\rho \\
\rho v_x \\
\rho v_y \\
\rho v_z \\
\rho v_x v_x + P_{xx} \\
\rho v_x v_y + P_{xy} \\
\rho v_x v_z + P_{xz} \\
\rho v_y v_y + P_{yy} \\
\rho v_y v_z + P_{yz} \\
\rho v_z v_z + P_{zz}
\end{pmatrix}
\]

\[
F = \begin{pmatrix}
\rho v_x \\
\rho v_x^2 + P_{xx} \\
\rho v_x v_y + P_{xy} \\
\rho v_x v_z + P_{xz} \\
3v_x P_{xx} + \rho v_x v_x v_z \\
(v_x P_{xy} + v_x P_{yy} + v_y P_{xx}) + \rho v_x v_x v_y \\
(v_x P_{xz} + v_x P_{zz} + v_z P_{xx}) + \rho v_x v_x v_z \\
(v_x P_{yy} + v_y P_{xy} + v_y P_{yy}) + \rho v_x v_y v_y \\
(v_x P_{yz} + v_y P_{yz} + v_z P_{yy}) + \rho v_x v_y v_z \\
(v_x P_{zz} + v_z P_{zz} + v_z P_{zz}) + \rho v_x v_z v_z
\end{pmatrix}
\]

\[
G = \begin{pmatrix}
\rho v_y \\
\rho v_y v_x + P_{xy} \\
\rho v_y v_y + P_{yy} \\
\rho v_y v_z + P_{yz} \\
(v_x P_{xy} + v_y P_{yy} + v_x P_{xy}) + \rho v_y v_x v_y \\
(v_x P_{yy} + v_y P_{xy} + v_y P_{yy}) + \rho v_y v_y v_y \\
(v_x P_{yz} + v_y P_{yz} + v_z P_{yy}) + \rho v_y v_z v_y \\
3v_y P_{yy} + \rho v_y v_y v_y \\
(v_y P_{yz} + v_y P_{yz} + v_y P_{yy}) + \rho v_y v_y v_y \\
(v_x P_{yz} + v_y P_{yz} + v_z P_{yy}) + \rho v_y v_z v_y \\
(v_y P_{yz} + v_y P_{yz} + v_z P_{yy}) + \rho v_y v_z v_y \\
(v_x P_{zz} + v_z P_{zz} + v_z P_{zz}) + \rho v_y v_z v_z
\end{pmatrix}
\]

\[
S = \begin{pmatrix}
0 \\
n(E_x + v_y B_x - v_z B_y) \\
n(E_y + v_z B_x - v_x B_z) \\
n(E_z + v_x B_y - v_y B_z) \\
2n v_x E_x + 2(q/m)(B_x P_{xy} + B_y P_{xx}) \\
n(v_x B_y + v_y E_x) + (q/m)(B_x P_{xy} - B_y P_{yx} + B_z P_{xx} - B_x P_{zz}) \\
n(v_x E_y + v_y E_x) + (q/m)(B_x P_{yz} + B_y P_{xz} - B_y P_{zz} - B_x P_{xy}) \\
2n v_y E_y + 2(q/m)(B_y P_{yz} - B_z P_{zy}) \\
n(v_y E_z + v_z E_y) - (B_y P_{xy} - B_z P_{xz} + B_x P_{zz} - B_x P_{yy}) \\
2n v_z E_z + 2(q/m)(B_z P_{yx} - B_x P_{yz})
\end{pmatrix}
\]
Multi-Fluid equations are solved together with Maxwell equations which are written as wave-equations (remember the first lecture, here in CGS-system):

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = -4\pi \rho \quad \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{A} = -\frac{4\pi}{c} \vec{J}
\]

Formally we combine these equations to:

\[
\frac{\partial^2 P}{\partial t^2} - \nabla^2 \vec{P} = \vec{S} \quad \text{where } P = (\phi, \vec{A}), \text{ and } S = (\rho, \vec{J})
\]

Equations are solved as a system of first order equations:

\[
\frac{\partial \vec{P}}{\partial t} = \vec{U} \quad \frac{\partial \vec{U}}{\partial t} = \vec{R} \quad \text{where } \vec{R} = \nabla^2 \vec{P} + \vec{S}.
\]
We have to solve consistently

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S
\]

\[
\frac{\partial \tilde{P}}{\partial t} = \tilde{U} \quad \frac{\partial \tilde{U}}{\partial t} = \tilde{R}
\]

Multi-Fluid equations

Maxwell equations

Laplace is discretized with 4th order 25-point stencil
(In earlier examples we used a 2th order 9-point stencil)

\[
\nabla^2 P = \frac{1}{\Delta x^2} \begin{pmatrix} c_1 & c_2 & c_3 & c_2 & c_1 \\ c_2 & c_4 & c_5 & c_4 & c_2 \\ c_3 & c_5 & c_6 & c_5 & c_3 \\ c_2 & c_4 & c_5 & c_4 & c_2 \\ c_1 & c_2 & c_3 & c_2 & c_1 \end{pmatrix} + O(h^4)
\]

\[
c_1 = 0, c_2 = -\frac{1}{30}, c_3 = -\frac{1}{60}, c_4 = \frac{4}{15}, c_5 = \frac{13}{15}, c_6 = -\frac{21}{5}
\]
Numerical scheme: Lax-Wendroff + slope limiter

- Method based on Lax-Wendroff scheme
- Additional feature: Non-oscillatory reconstruction near gradients.

Predictor step:

\[ w^{n+1/2}_i = w^n_i - \frac{\lambda}{2} F^x (w^n_i) \]

Corrector step:

\[ w^{n+1}_{i+1/2} = \frac{1}{2} \left( w^n_i + w^n_{i+1} \right) + \frac{1}{8} \left( w^x_i - w^x_{i+1} \right) - \frac{\lambda}{2} \left[ F(w^{n+1/2}_{i+1}) - F(w^{n+1/2}_i) \right] \]
Test problems: Weibel instability

- The Weibel instability is driven in a collisionless plasma by the anisotropy of the particle velocity distribution function of the plasma
  - Shocks
  - Strong temperature gradient
- Magnetic fields are generated so that the distribution function becomes isotropic

Initial electron temperature is anisotropy $T_{zz} = 10T_{xx}$, ions are isotropic. Ion mass is $M_i = 25M_e$. The simulation is performed on a 2D domain ($N_x = N_y = 128$). Periodic boundary conditions are adopted in both coordinate directions.
$J_z$ (color-coded) and magnetic field lines, $t_{\omega_{pe}} = 188.1818$
Comments on Weibel instability development

- The process of instability development is accompanied by creation of localised current sheets, sustained by self-consistent magnetic fields. Currents with the same direction are attracted because of their magnetic field.

- Currents and magnetic fields increase through merger of currents due to magnetic field lines reconnection. This leads to decrease of temperature anisotropy.
Multi fluid simulations
Summary

• Solve coupled system of fluid and Maxwell equations.
• Uses first 10 moments of 6D-distribution functions.
• Written as first order in time system.
• Flux-conservative part + Source-term.
• Based on Lax-Wendroff scheme.
• Slope-limiter to avoid spurious oscillations near strong gradients.
• Tested with Weibel instability in anisotropic plasma.
I am grateful to all people who helped me to prepare this lecture by providing material, discussions and checking lecture notes and exercises:

• Nina Elkina
• Julia Thalmann
• Tilaye Tadesse
• Elena Kronberg
• Many unknown authors of Wikipedia and other online sources.

THANK YOU
For this lecture I took material from

- Wikipedia and links from Wikipedia
- Numerical recipes in C, Book and
  http://www.fizyka.umk.pl/nrbook/bookcpdf.html
- Lecture notes *Computational Methods in Astrophysics*
  http://compschoolsolaire2008.tp1.ruhr-uni-bochum.de/
- Presentation/Paper from Nina Elkina
- MHD-equations in conservative form:
  http://www.lsw.uni-heidelberg.de/users/sbrinkma/seminar051102.pdf
THE END