Welcome to the

Lecture on stellar atmospheres

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Germany

Stellar Atmospheres

Outline:

- Introduction
- Radiation field
- Radiation transfer
- Emission and absorption
- Radiative equilibrium
- Hydrostatic equilibrium
- Stellar atmosphere models

Stellar Atmospheres: Literature

- Dimitri Mihalas
  - Stellar Atmospheres, W.H. Freeman, San Francisco
- Albrecht Unsöld
  - Physik der Sternatmosphären, Springer Verlag (in German)
- Rob Rutten
  - Lecture Notes Radiative Transfer in Stellar Atmospheres http://www.fys.ruu.nl/~rutten/node20.html

Why physics of stellar atmospheres?

Physics

Stellar atmospheres as laboratories

Plasma-, atomic-, and molecular physics, hydrodynamics, thermodynamics

Basic research

Technical application

Astronomy

Spectral analysis of stars

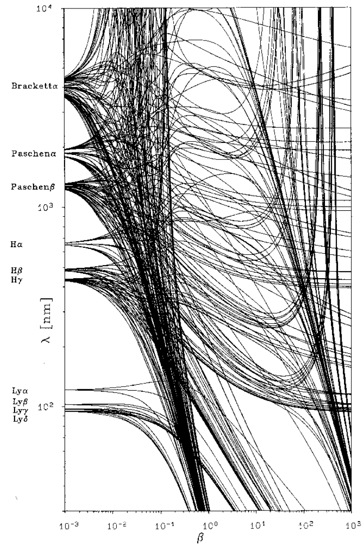
Structure and evolution of stars

Galaxy evolution

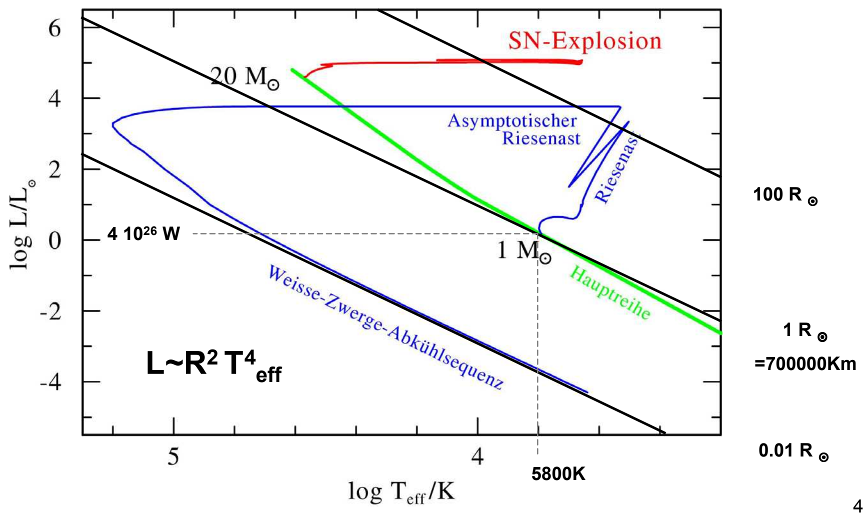
Evolution of the Universe

Magnetic fields in white dwarfs and neutron stars

Shift of spectral lines with increasing field strength

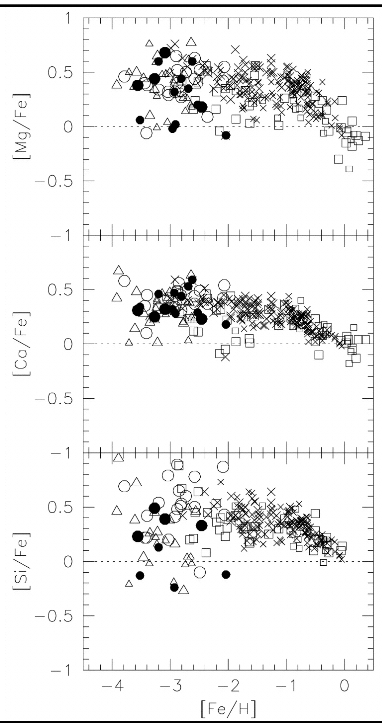


Hertzsprung Russell Diagram



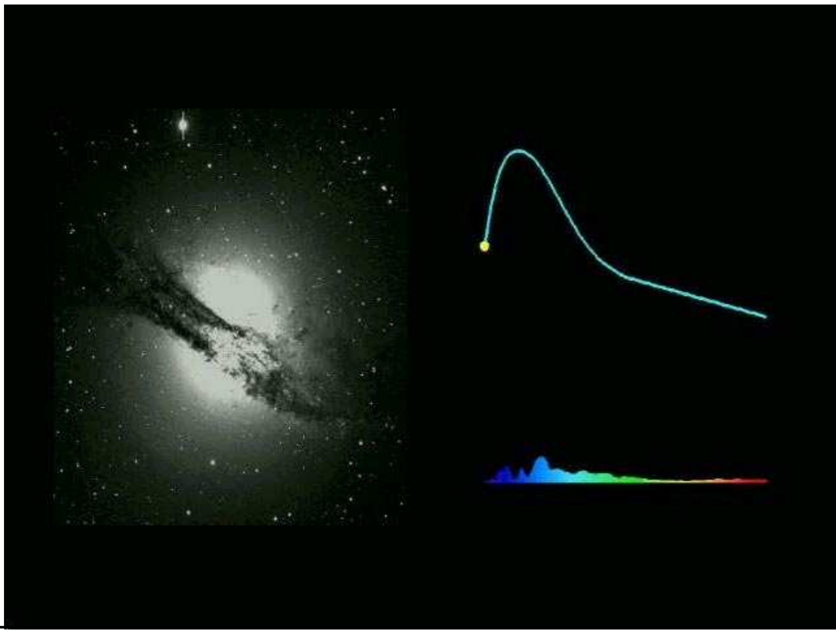
Stellar Atmospheres: Motivation

Chemical evolution of the Galaxy

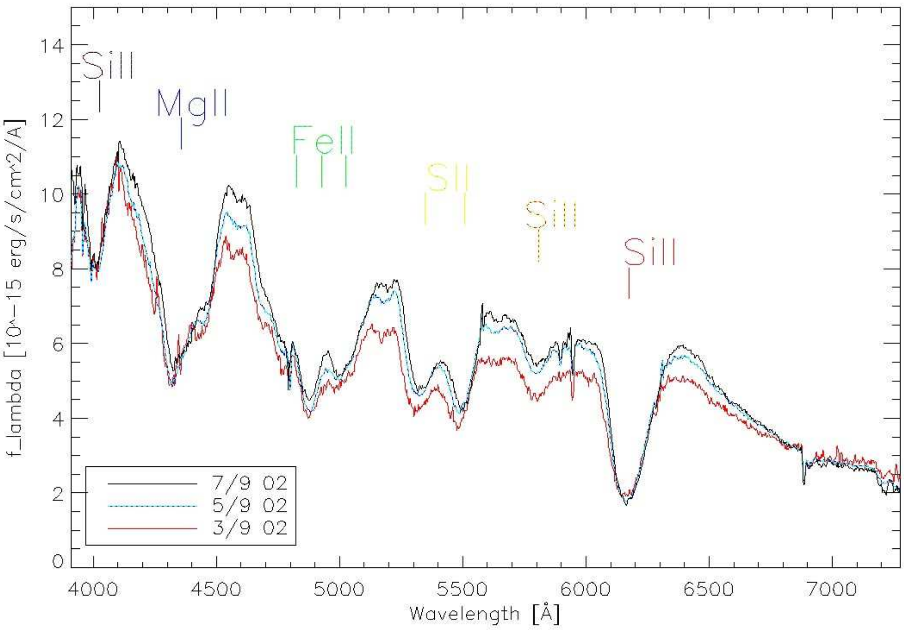


Carretta et al.
2002, AJ 124, 481

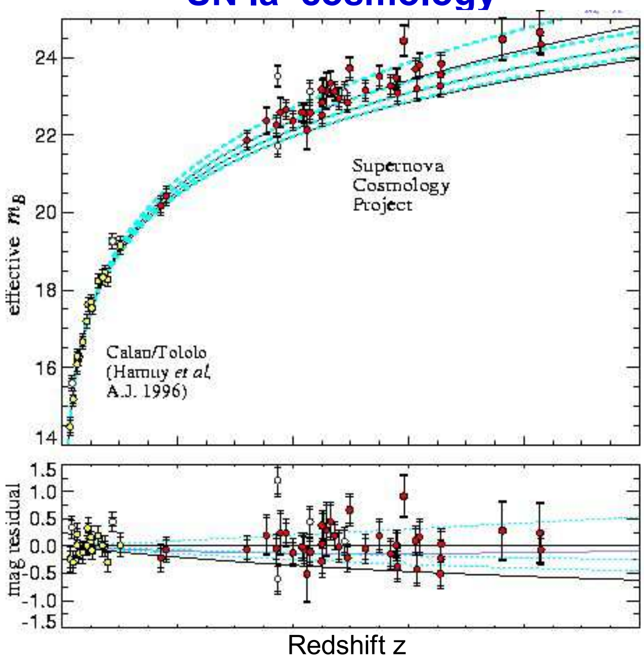
SN movie



SN Ia



SN Ia cosmology



Ω_M, Ω_Λ

0 , 1

0.5, 0.5

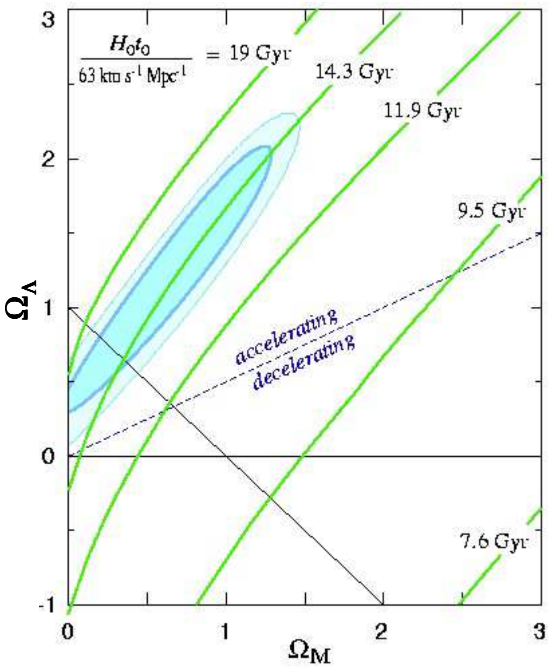
1 , 0

1.5, -.5

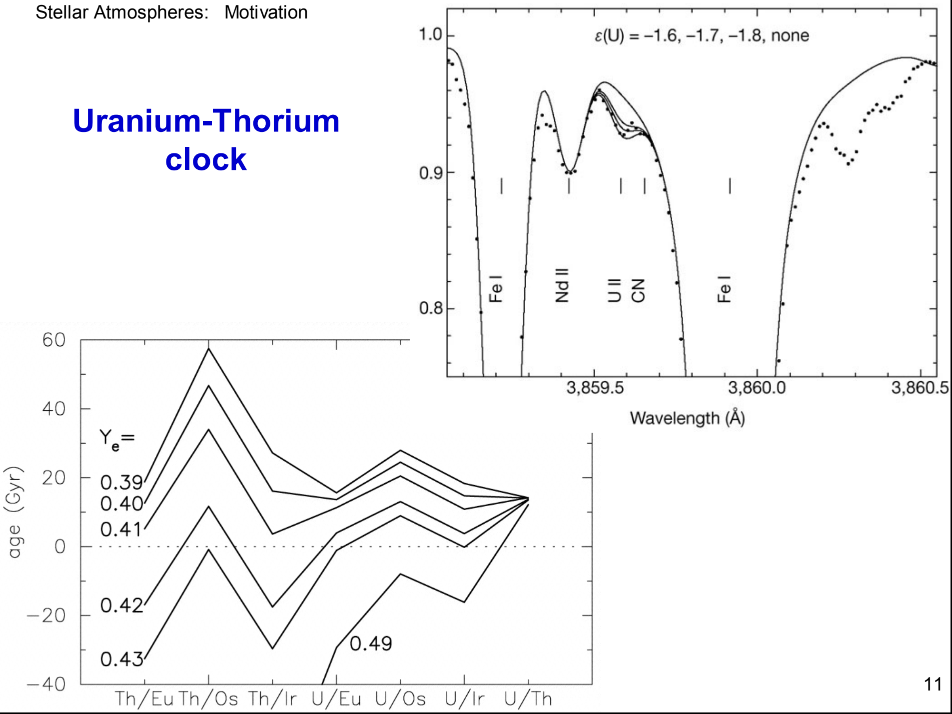
0 , 0

1 , 0

2 , 0



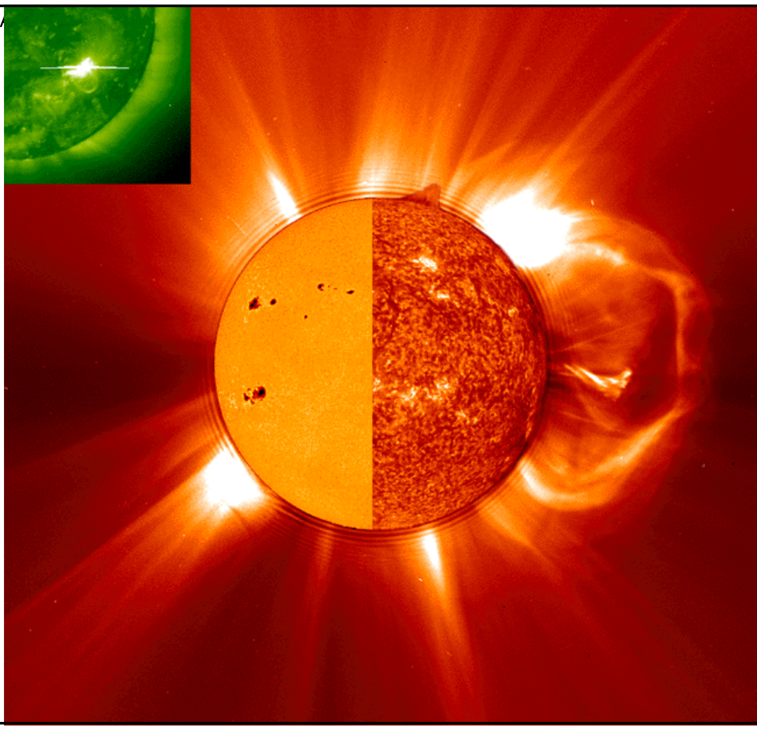
Uranium-Thorium clock



Stellar atmosphere – definition

- From outside visible, observable layers of the star
- Layers from which radiation can escape into space
  - Dimension
- Not stellar interior (optically thick)
- No nebula, ISM, IGM, etc. (optically thin)
- But: chromospheres, coronae, stellar winds, accretion disks and planetary atmospheres are closely related topics

Stellar



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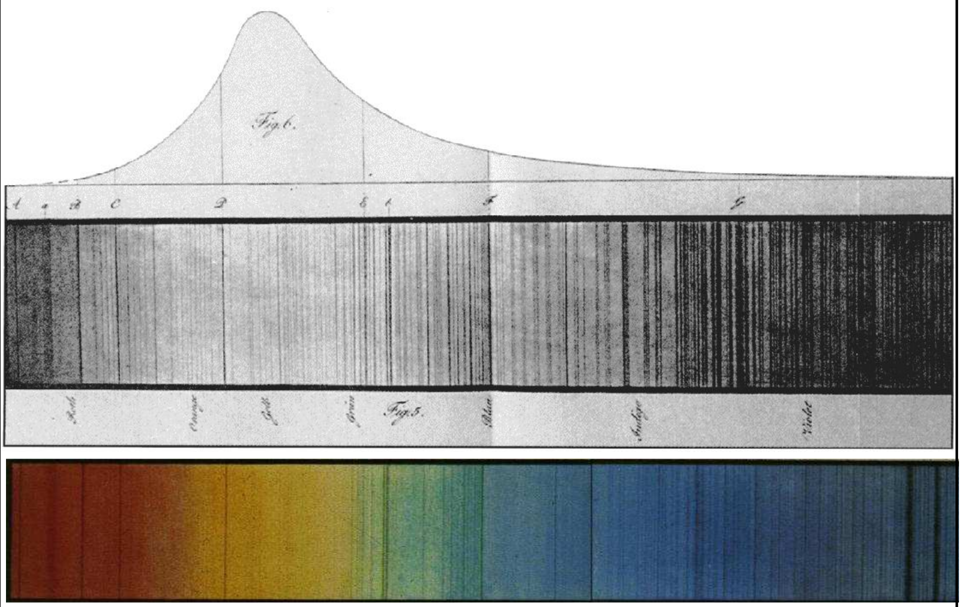
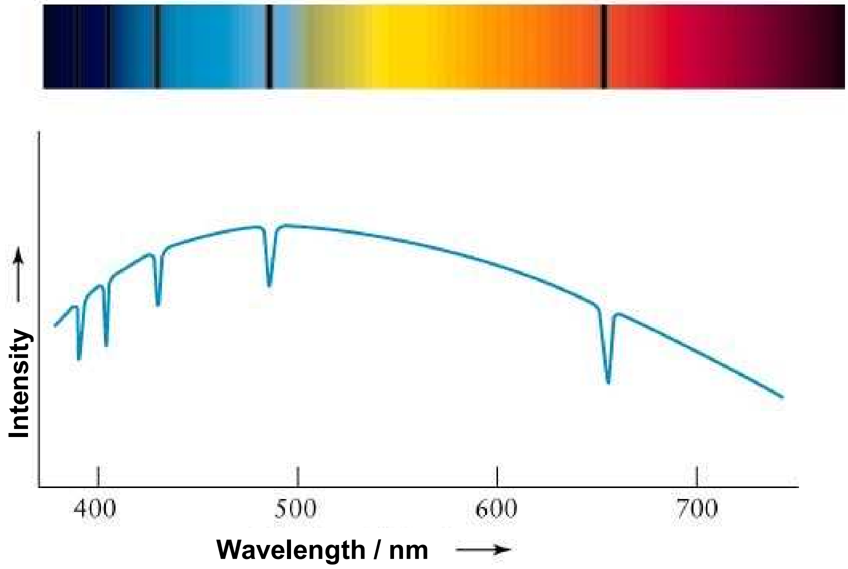
Stellar Atmospheres: Motivation

Fraunhofer lines



Spectrum - schematically



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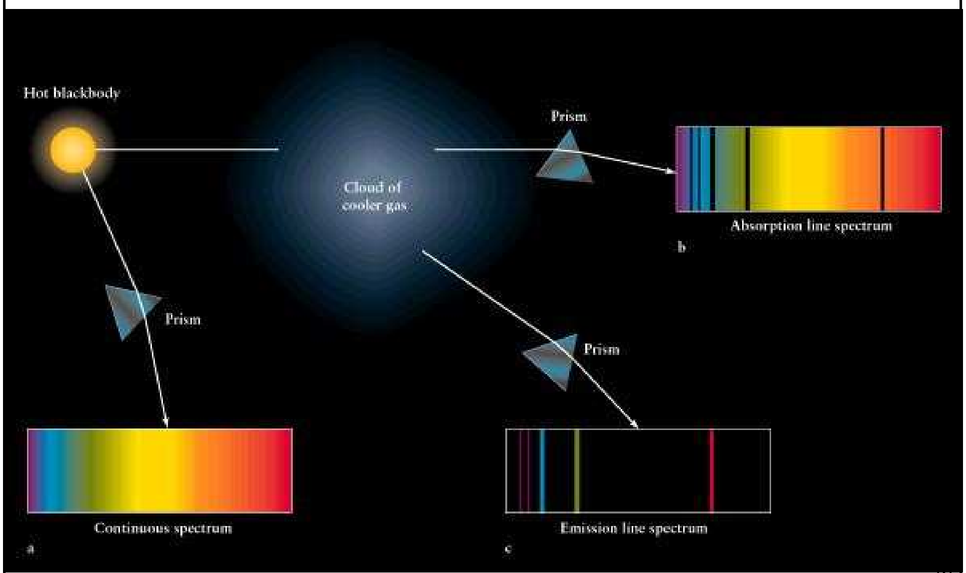
Spectrum formation



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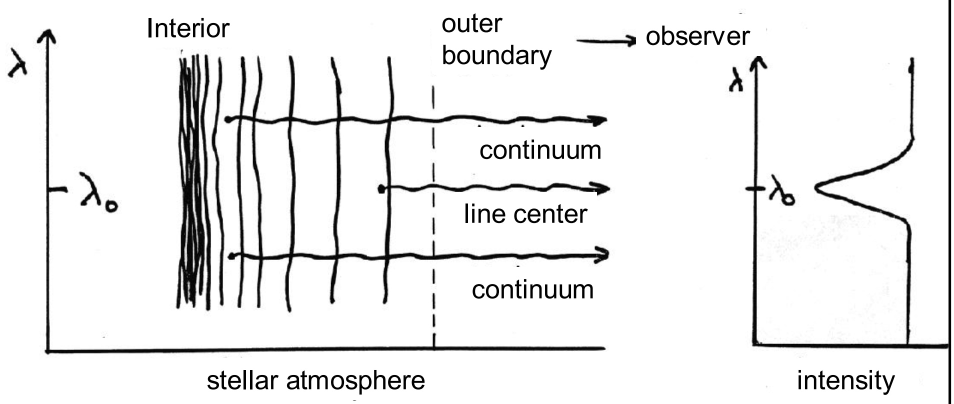
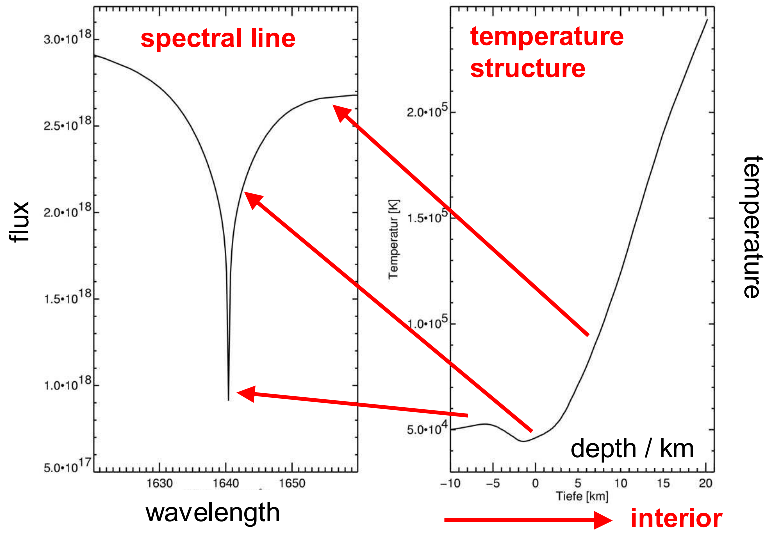
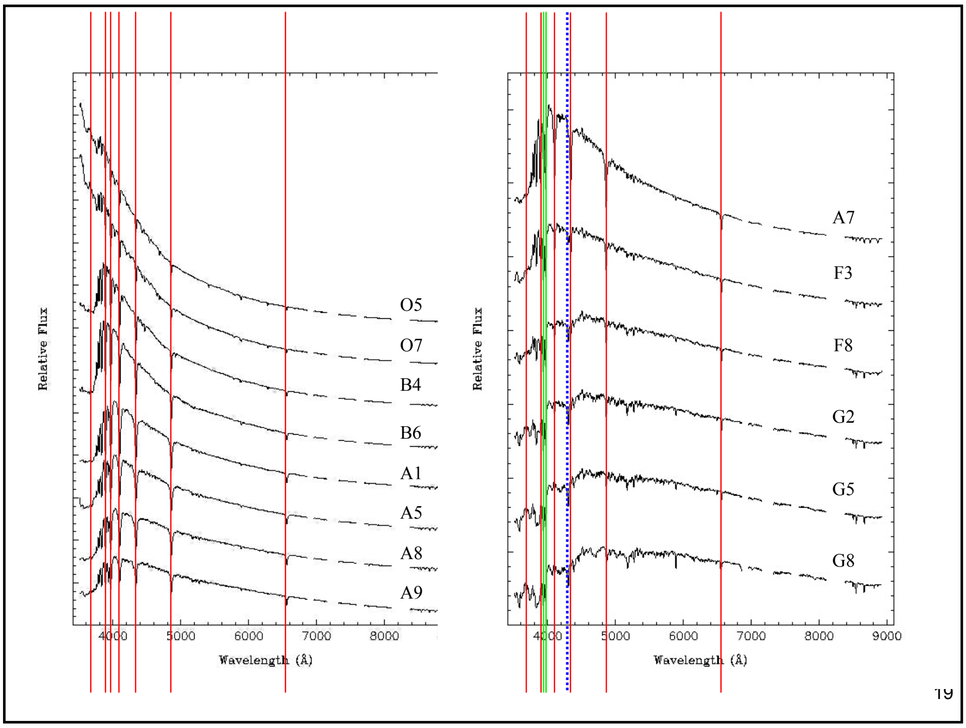
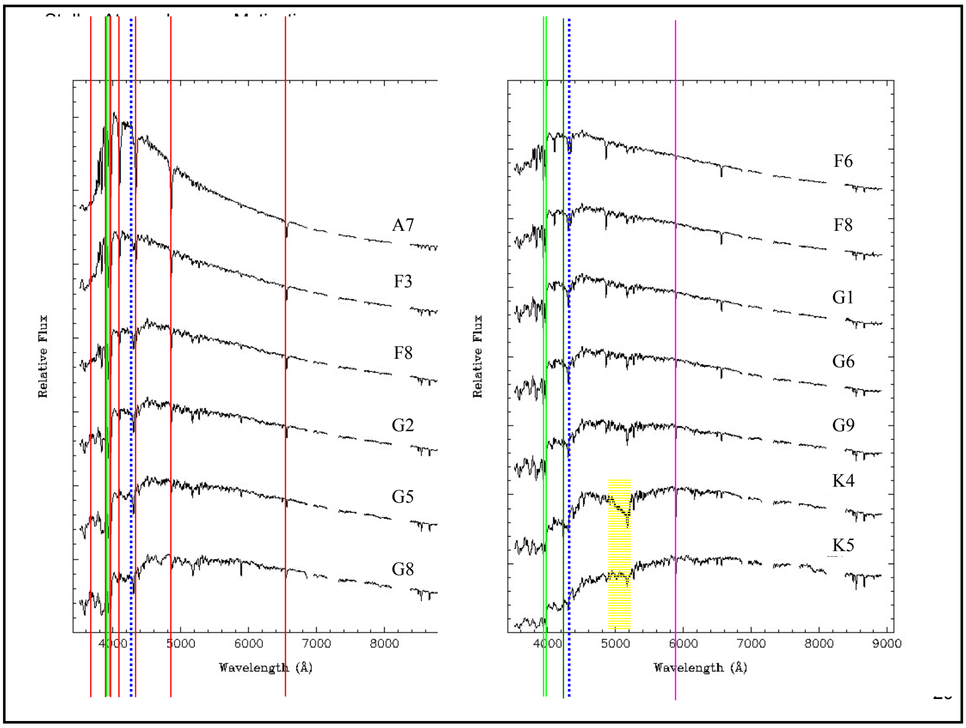
Formation of absorption lines



Line formation / stellar spectral types




Relative Flux
O5
O7
B4
B6
A1
A5
A8
A9
4000
5000
6000
7000
8000
Wavelength (Å)
A7
F3
F8
G2
G5
G8
9000
19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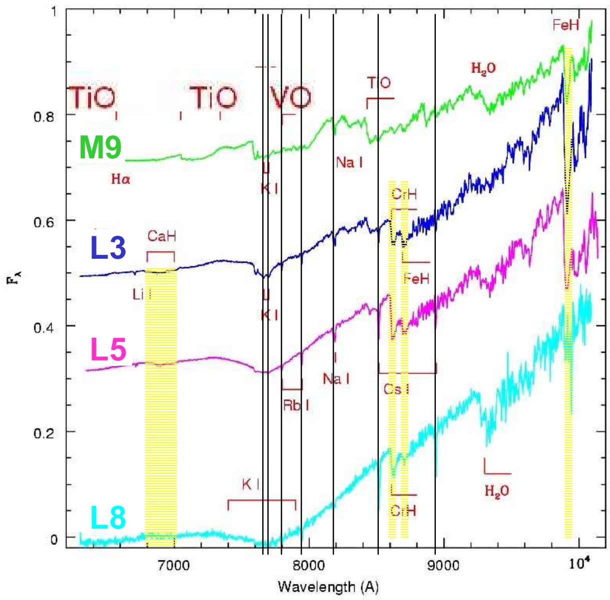
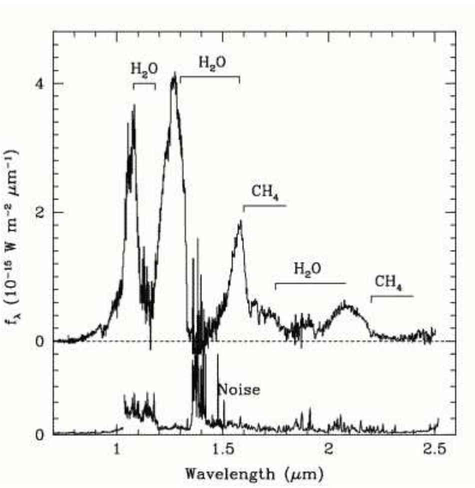


Classification scheme

T dwarfs



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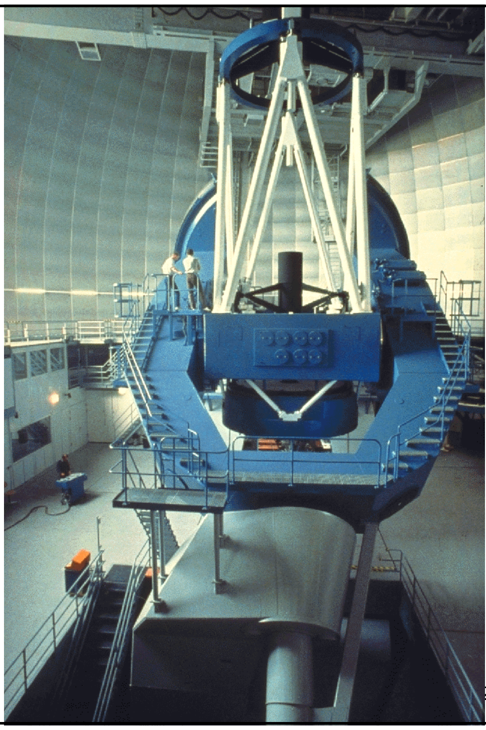
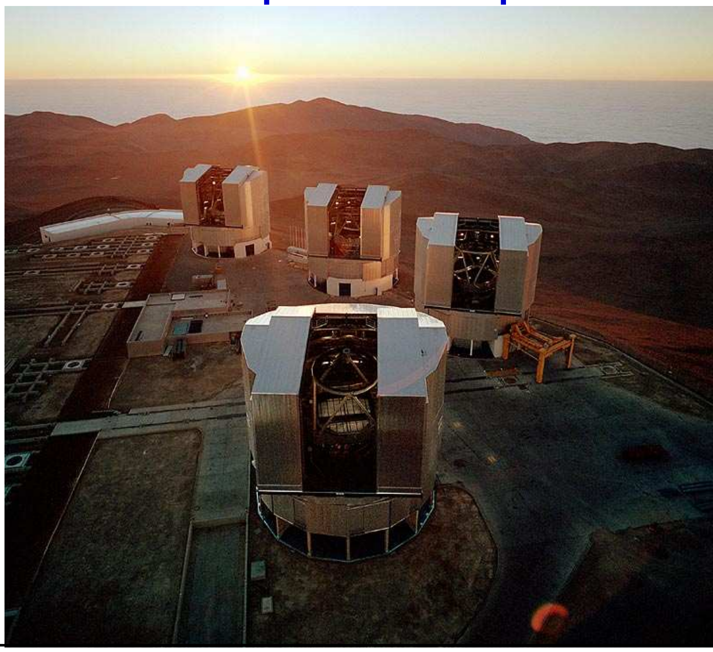
Optical telescopes



Calar Alto (Spain)
3.5m telescope

Stellar Atmospheres: Motivation

Optical telescopes

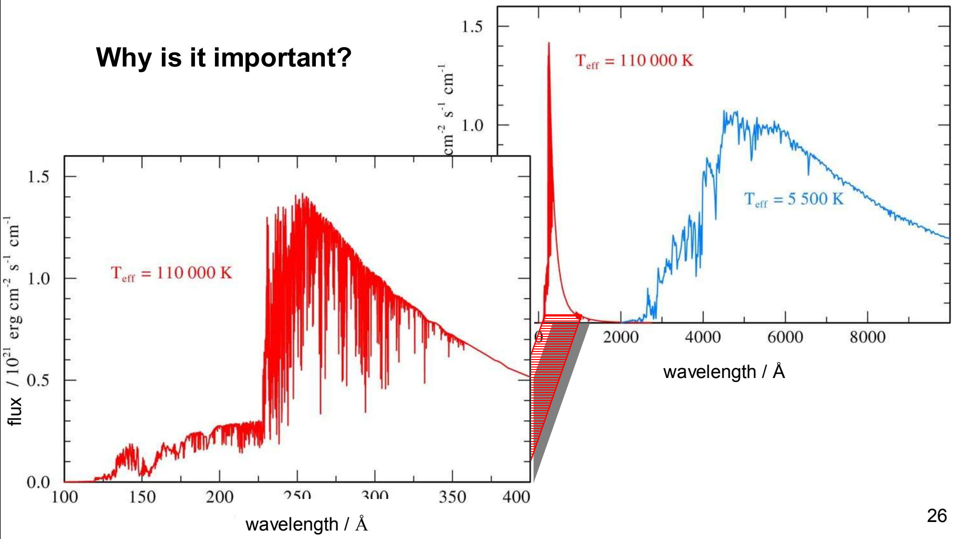


ESO/VLT

Stellar Atmospheres: Motivation

UV / EUV observations

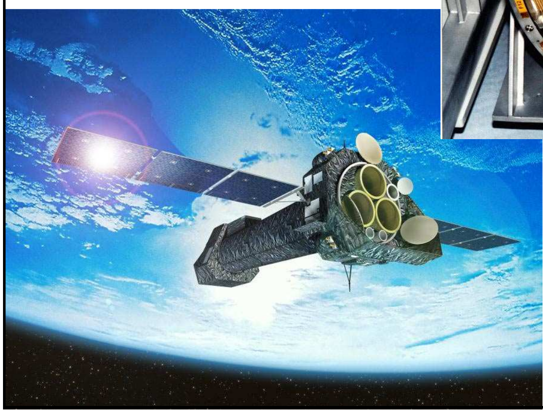
Why is it important?



UV/optical telescopes

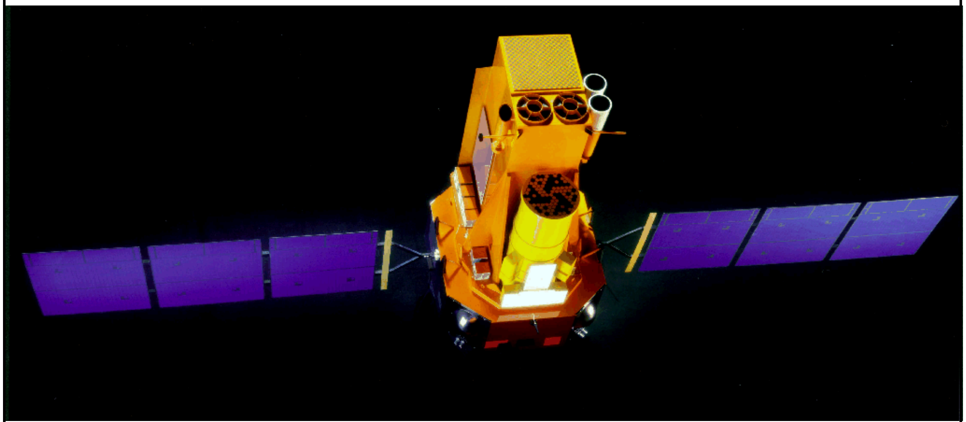
HST

X-ray telescopes



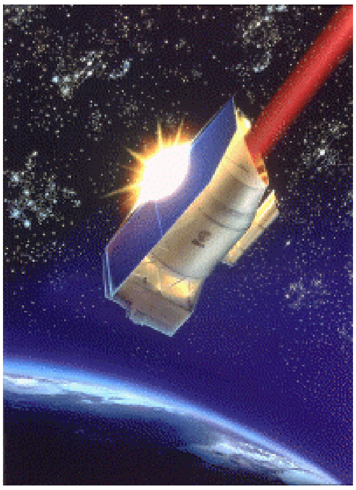
XMM

Gamma-ray telescopes



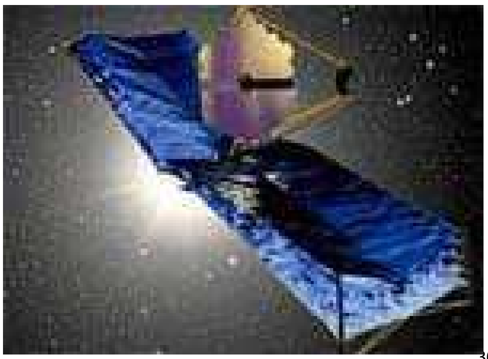
INTEGRAL

Infrared observatories

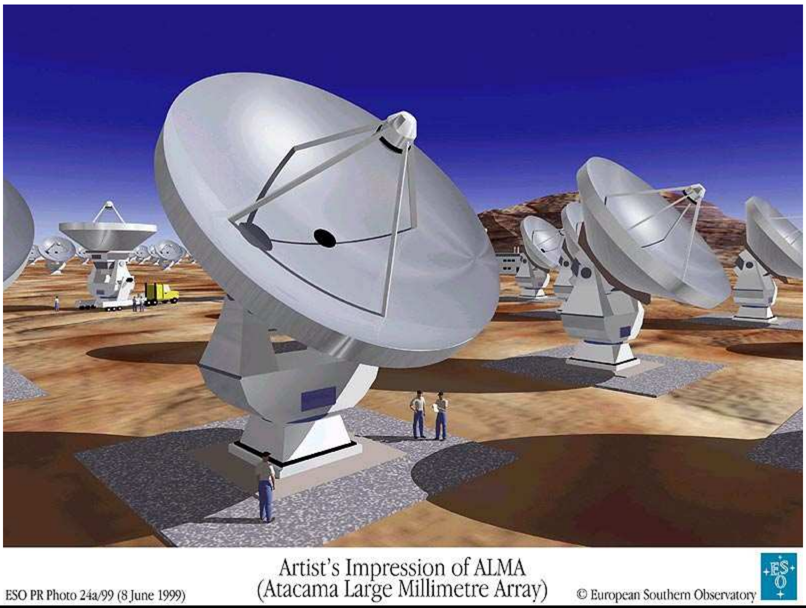


ISO

JWST

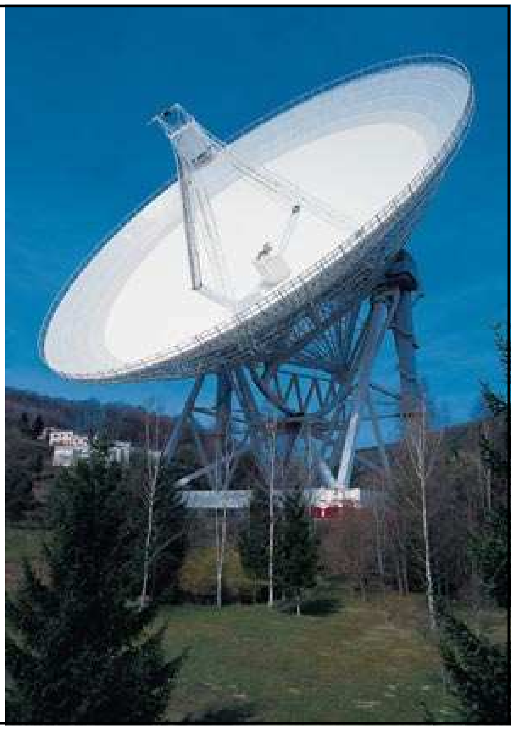


Sub-mm telescopes



Radio telescopes

100m dish at Effelsberg



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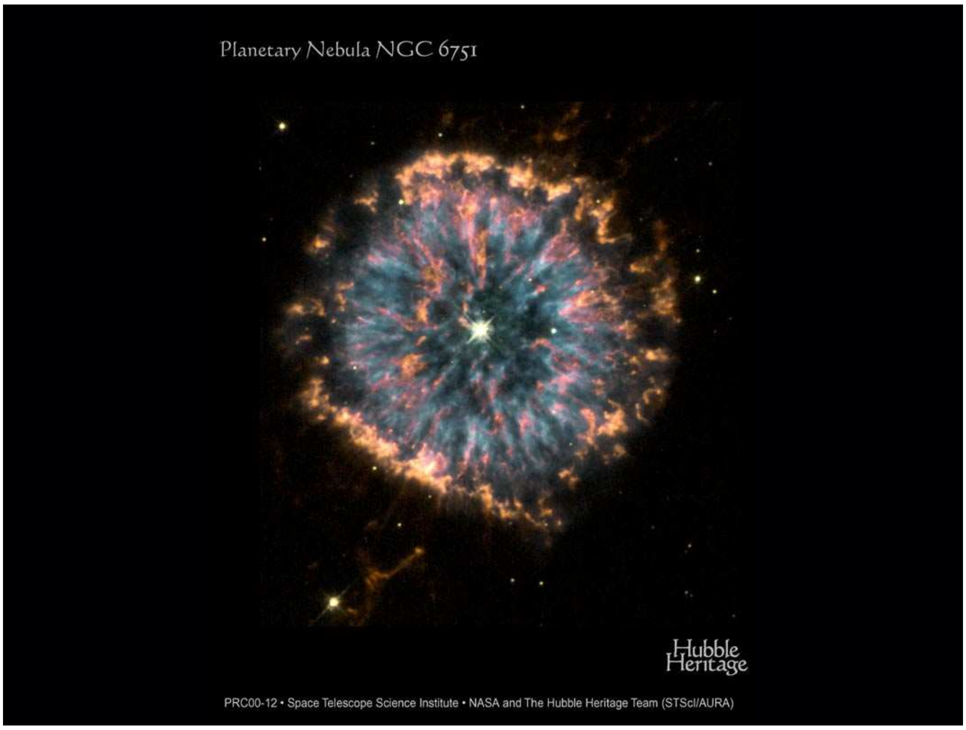
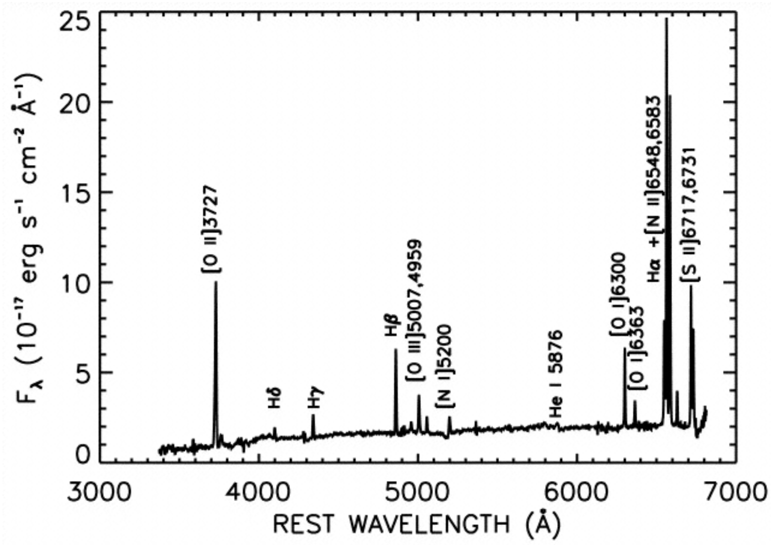
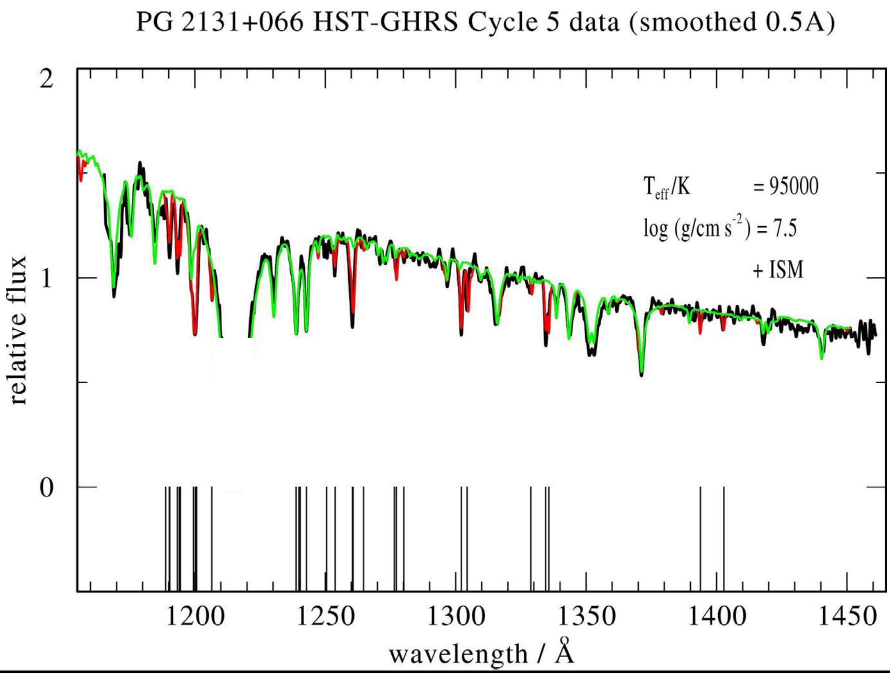
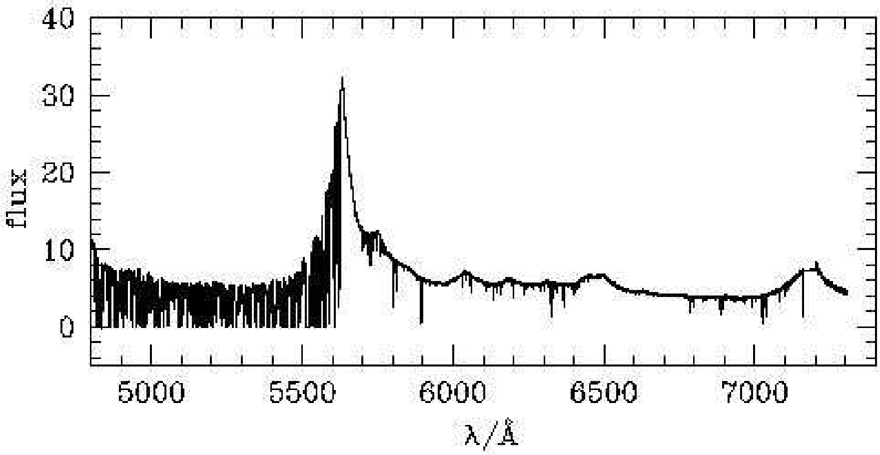


Planetary nebula spectrum





Quasar + IGM spectrum



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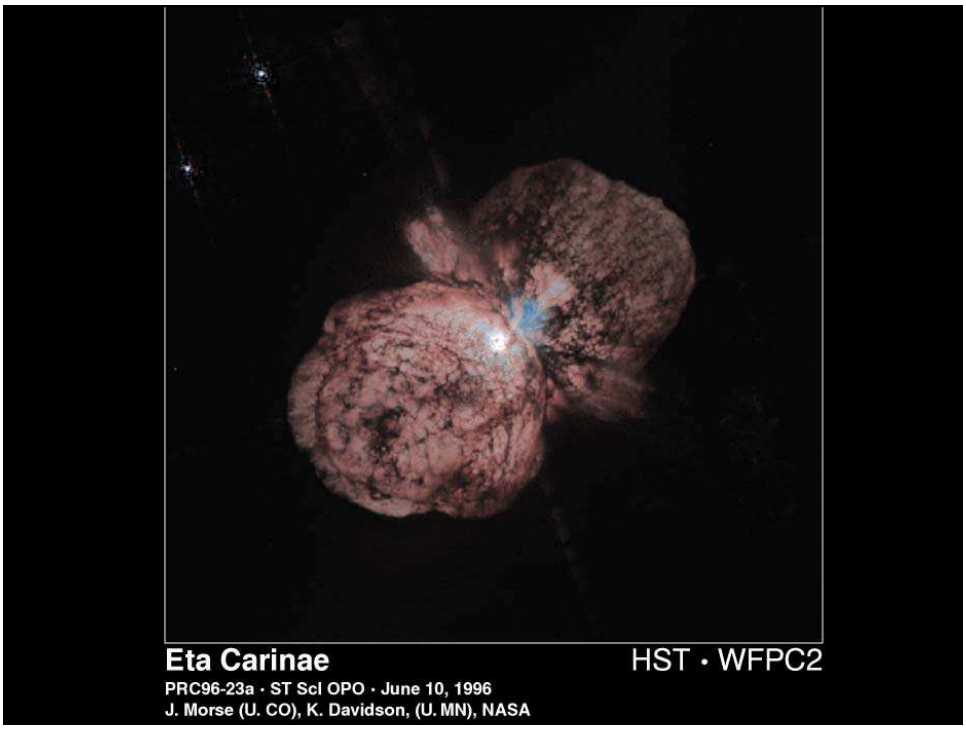
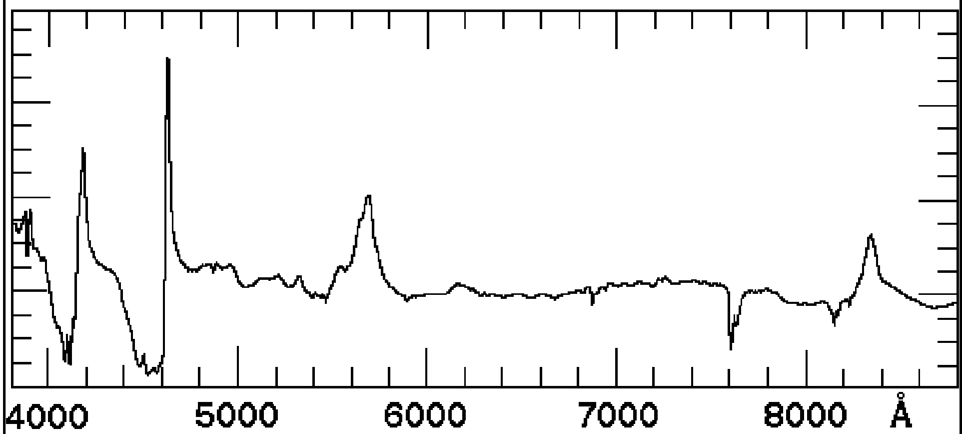
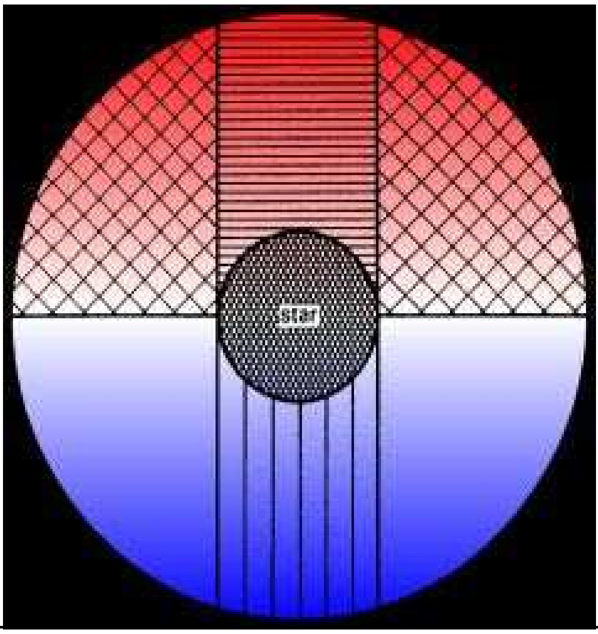


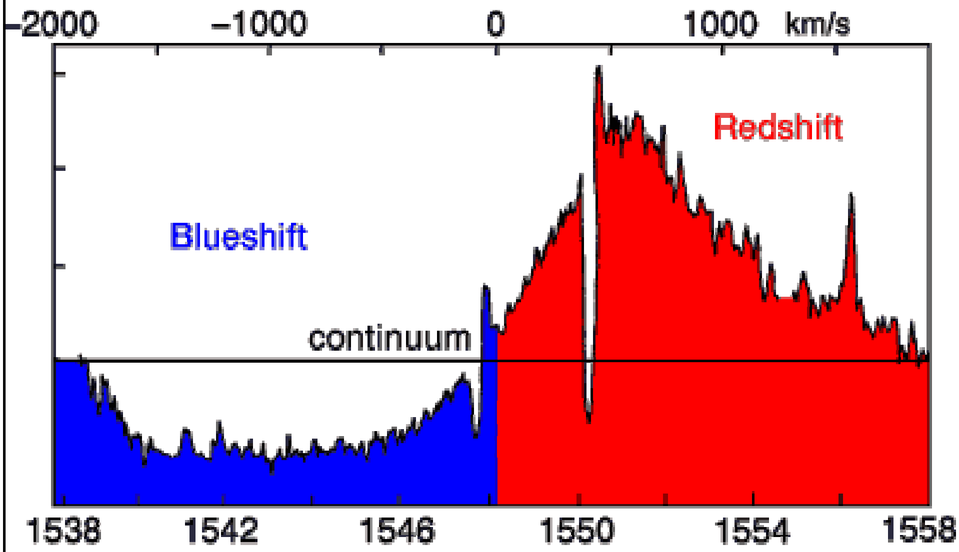
Stellar wind spectrum



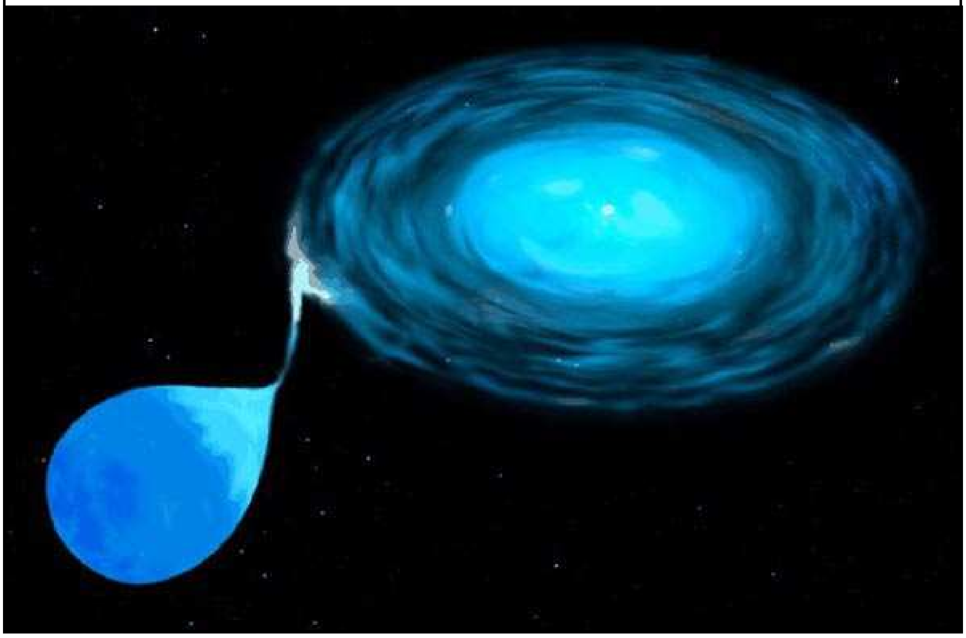
Formation of wind spectrum (P Cygni line profiles)



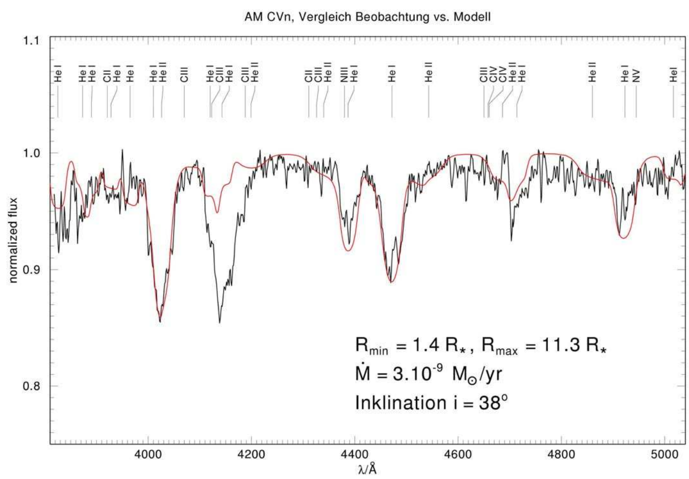
Stellar winds – P Cyg profiles



Accretion disks

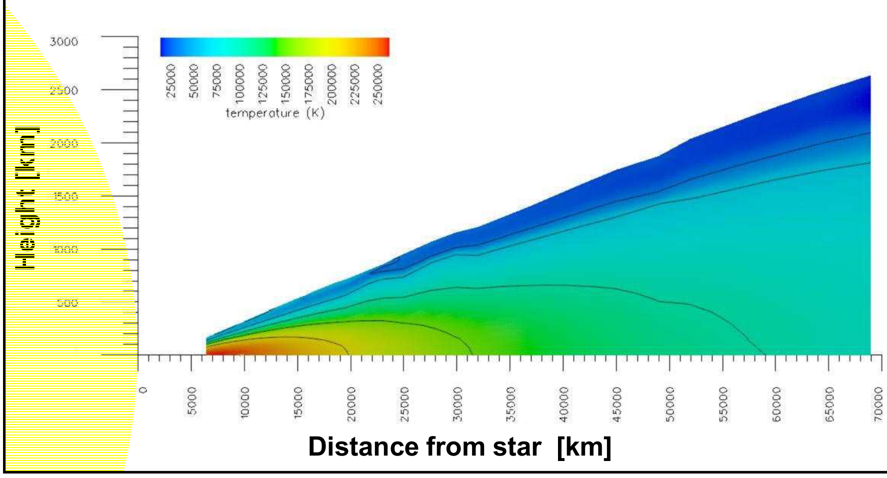


AM CVn disk spectrum

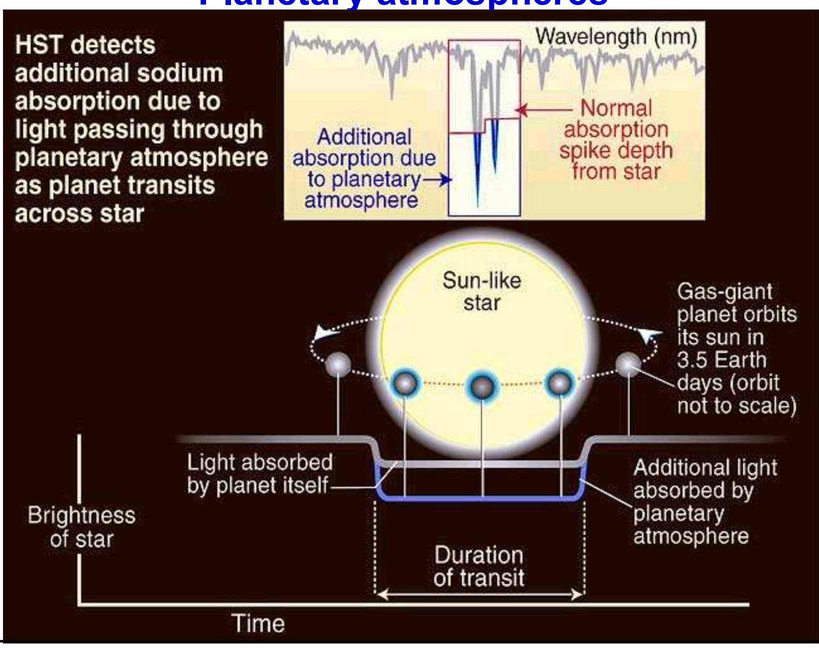


models

Temperature structure of an accretion disk



Planetary atmospheres



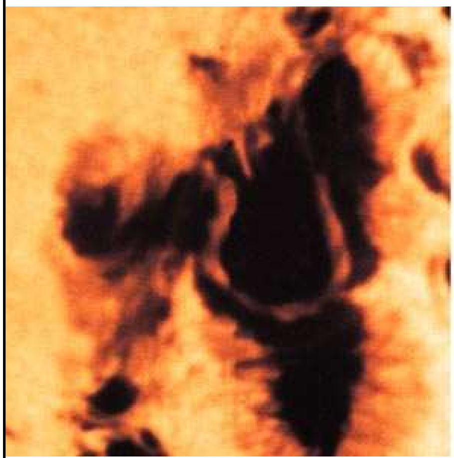
Quantitative spectral analyses – what can we learn?

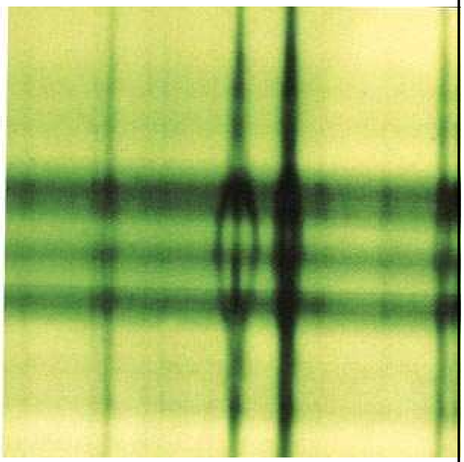
Shape of line profile:
- Temperature Film
- Density Film
- Abundance Film
- Rotation
- Turbulence
- Magnetic field

Line position:
- Chemical composition
- Velocities
- Redshift

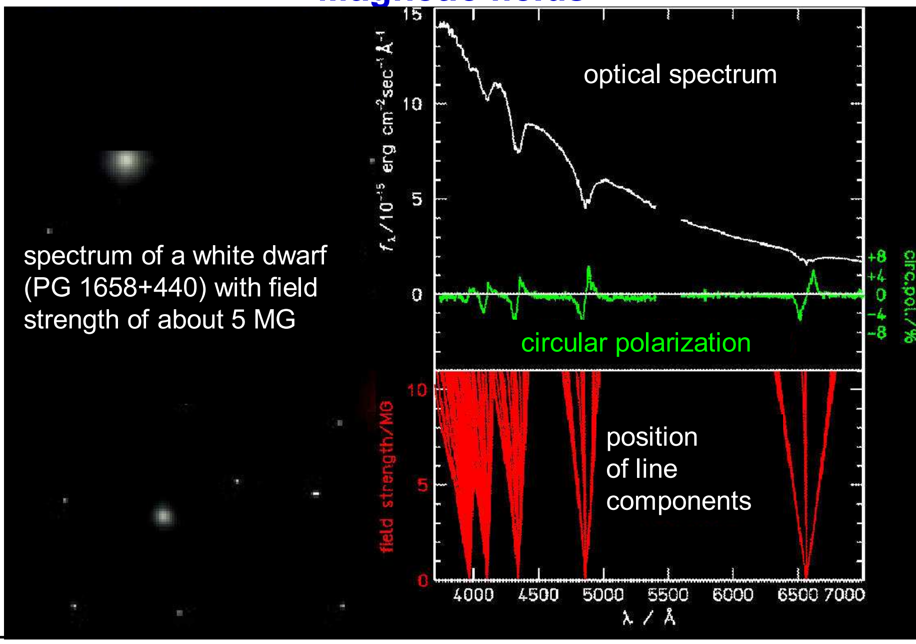
Temporal variation:
- Companion
- Surface structure
  - Spots
  - Pulsation

Zeeman effect

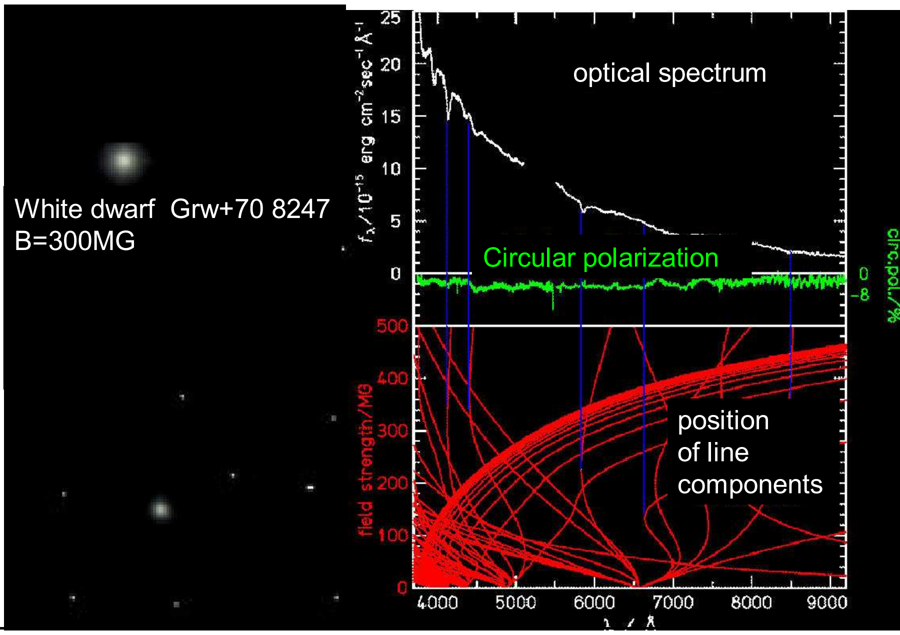




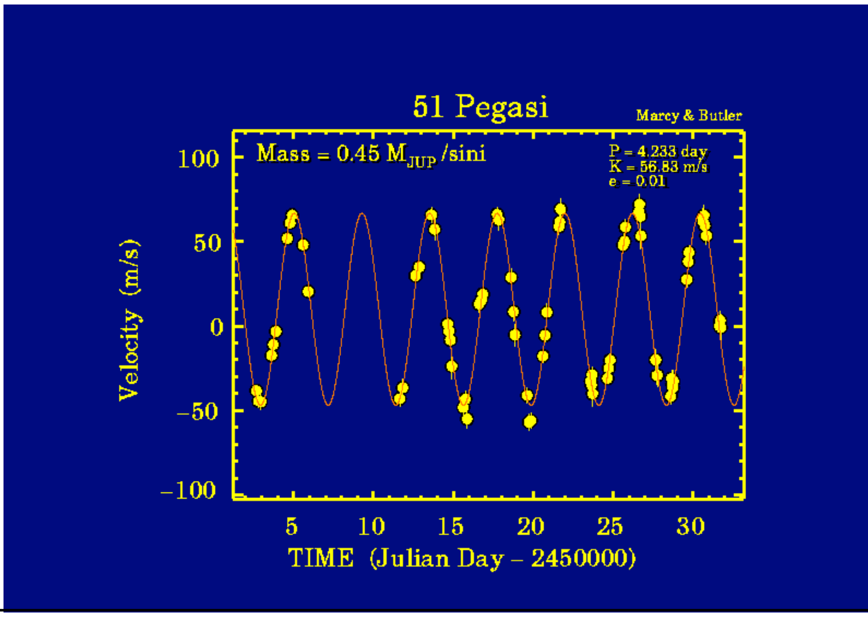
Magnetic fields

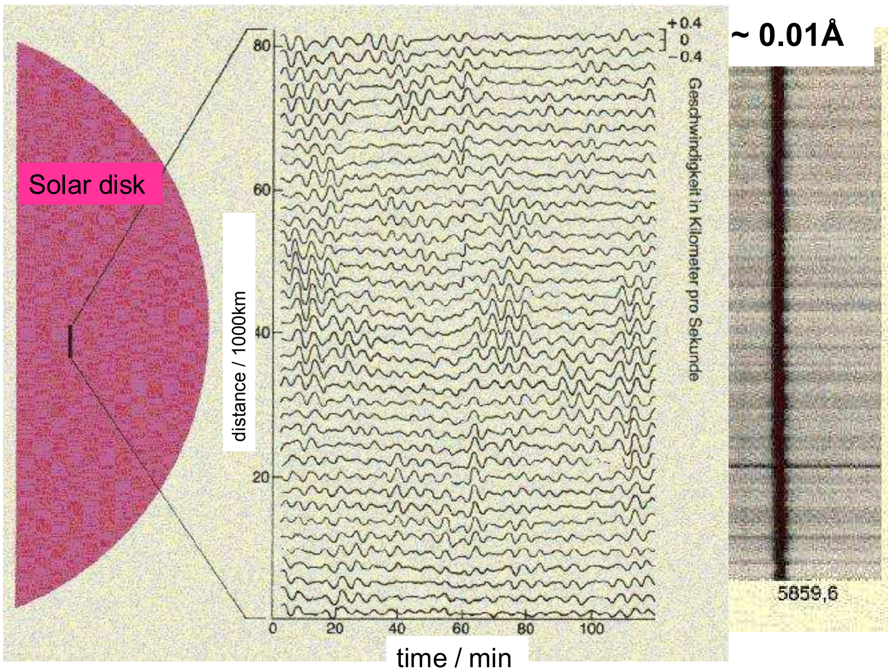


Magnetic fields

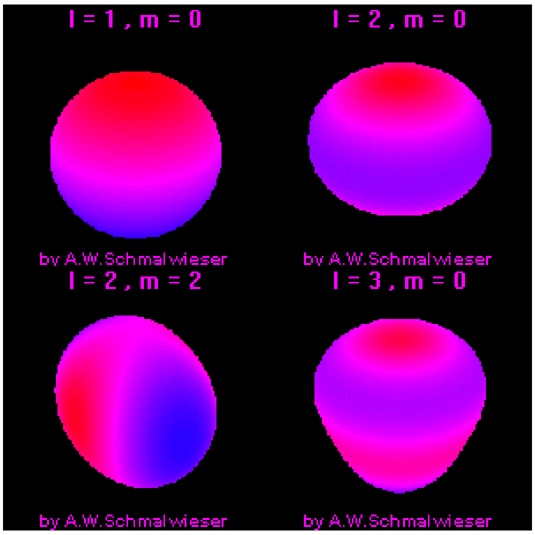


Extrasolar planets

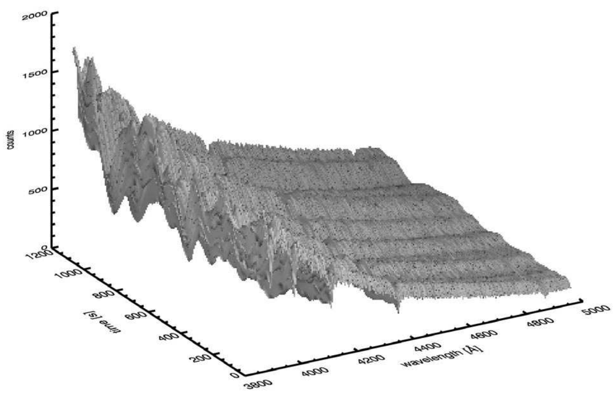


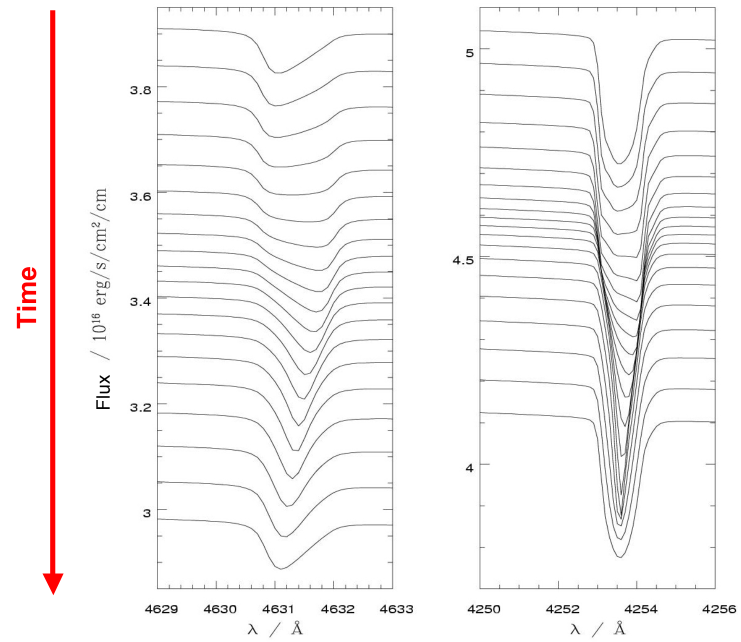


Non-radial pulsation modes

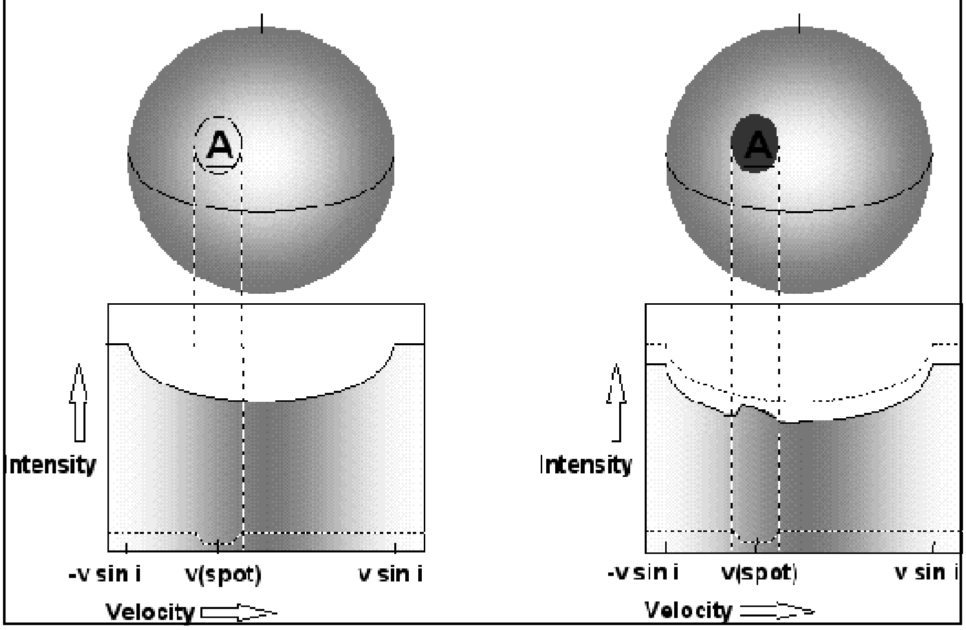


Time resolved spectroscopy





Doppler tomography



Summary – stellar atmospheres theory

The atmosphere of a star contains less than one billionth of its total mass, so, **why do we care at all?**

- The atmosphere of a star is that what we can **see**, measure, and analyze.
- The stellar atmosphere is therefore the source of information in order to put a star from the color-magnitude diagram (e.g. B-V,m_v) of the observer into the HRD (L,T_{eff}) of the theoretician and, hence, to drive the **theory of stellar evolution**.
- Atmosphere analyses reveal element abundances and show us results of **cosmo-chemistry**, starting from the earliest moments of the formation of the Universe.
- Hence, working with stellar atmospheres enables a test for **big-bang theory**.
- Stars are the **building blocks of galaxies**. Our understanding of the most distant (hence most early emerged) galaxies, which cannot be resolved in single stars, is not possible without knowledge of processes in atmospheres of single stars.
- Work on stellar atmospheres is a big challenge. The atmosphere is that region, where the transition between the thermodynamic equilibrium of the stellar interior into the empty blackness of space occurs. It is a region of extreme **non-equilibrium states**.

Summary – stellar atmospheres theory

Important source of information for many disciplines in astrophysics

- research for pure knowledge, contribution to our culture
- ambivalent applications (e.g. nuclear weapons)

Application of diverse disciplines

- physics
- numerical methods

Still a very active field of research, many unsolved problems

- e.g. dynamical processes

The Radiation Field

Description of the radiation field

Macroscopic description:

Specific intensity $$I_\nu(\nu,\vec{n},\vec{r},t)$$

as function of frequency, direction, location, and time; energy of radiation field (no polarization)

•in frequency interval (ν,ν+dν)

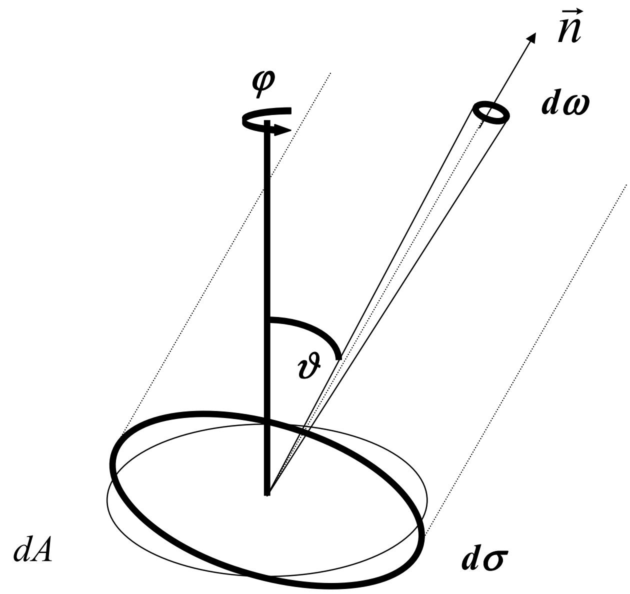
•in time interval (t,t+dt)

•in solid angle dω around n

•through area element dσ at location r $\perp$ n

$$\boxed{I_\nu(\nu,\vec{n},\vec{r},t) := \frac{d^4E}{d\nu\, dt\, d\omega\, d\sigma}}$$

The radiation field



Relation $I_\nu \leftrightarrow I_\lambda$

Energy in frequency interval (ν,ν+Δν) $\rightarrow I_\nu$
Energy in wavelength interval (λ,λ+Δλ) $\rightarrow I_\lambda$

i.e. $d^4E = I_\lambda\, dA \cos\vartheta\, dt\, d\lambda\, d\omega$

thus $I_\nu\,|d\nu| = I_\lambda\,|d\lambda|$

with $\nu\lambda = c \Rightarrow \frac{d\nu}{d\lambda} = -\frac{c}{\lambda^2}$ $\boxed{I_\nu = \frac{c}{\nu^2} I_\lambda \quad I_\lambda = \frac{c}{\lambda^2} I_\nu}$

	I_ν	I_λ
Dimension	$\frac{\text{energy}}{\text{area time freq. solid angle}}$	$\frac{\text{energy}}{\text{area time wavelength solid angle}}$
Unit	$\frac{\text{erg}}{\text{cm}^2\text{ s Hz sterad}}$	$\frac{\text{erg}}{\text{cm}^2\text{ s Å sterad}}$

Invariance of specific intensity

Irradiated energy:

$$dE = I_\nu(\nu,\vartheta) d\nu \cos\vartheta \, dA \, d\omega$$

dA' as seen from dA subtends solid angle $d\omega$

$$d\omega = \cos\vartheta' dA' / d^2$$

$$\rightarrow dE = I_\nu(\nu,\vartheta) d\nu \frac{\cos\vartheta \, dA \cos\vartheta' \, dA'}{d^2}$$

now, dA as seen from dA'

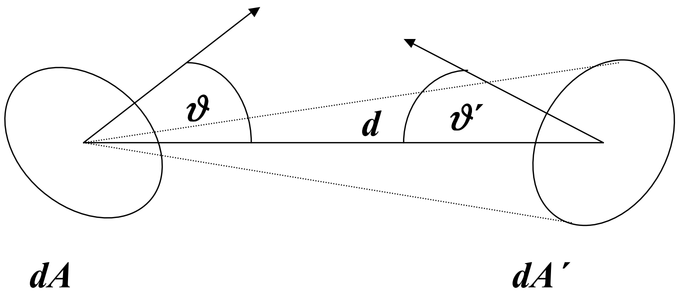
$$dE' = I'_\nu(\nu,\vartheta) d\nu \frac{\cos\vartheta \, dA \ \cos\vartheta' \, dA'}{d^2}$$

if no sources or sinks along d:

$$dE = dE' \Leftrightarrow I'_\nu = I_\nu$$

The specific intensity is distance independent if no sources or sinks are present.

Irradiance of two area elements



Specific Intensity

Specific intensity can only be measured from extended objects, e.g. Sun, nebulae, planets

Detector measures energy per time and frequency interval

$$dE = I_\nu \cos\vartheta \, d\omega \, A$$

e.g. A is the detector area

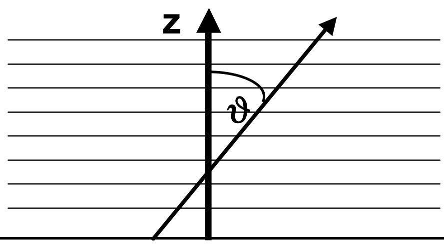
$d\omega \sim (1'')^2$ is the seeing disk

Special symmetries

- Time dependence unimportant for most problems
- In most cases the stellar atmosphere can be described in plane-parallel geometry

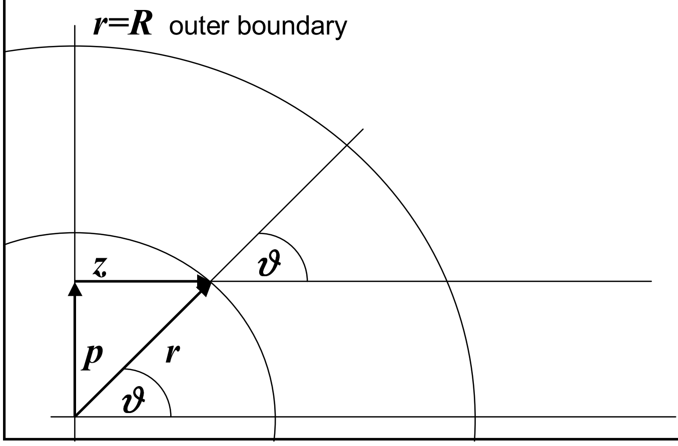
Sun: $\dfrac{\text{atmosphere}}{\text{radius}} = \dfrac{200\text{ km}}{700000\text{ km}} = \dfrac{1}{3500} << 1$

$$\mu := \cos\vartheta \quad I_\nu = I_\nu(\nu, \mu, z)$$



- For extended objects, e.g. giant stars (expanding atmospheres) spherical symmetry can be assumed

spherical coordinates: $I_\nu(\nu, \mu, r)$ Cartesian coordinates: $I_\nu(\nu, p, z)$



Integrals over angle, moments of intensity

- The 0-th moment, mean intensity

$$J_\nu = \frac{1}{4\pi} \oint_{4\pi} I_\nu(\vec{n}) d\omega \quad \text{with spherical coordinates}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} I_\nu \sin\vartheta \, d\vartheta \, d\phi \quad \text{with } \mu := \cos\vartheta$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} I_\nu \, d\mu \, d\phi$$

- In case of plane-parallel or spherical geometry

$$\boxed{J_\nu = \frac{1}{2} \int_{-1}^{1} I_\nu \, d\mu}$$

$$\frac{\text{energy}}{\text{area time frequency}} \qquad \frac{\text{erg}}{\text{cm}^2 \text{ s Hz}}$$

J_ν is related to the energy density u_ν

radiated energy through area element dA during time dt:

$$dE = I_\nu \, d\nu \; dt \; d\omega \; dA$$

$$l = c\,dt \Rightarrow dV = l\,dA = c\,dt\,dA$$

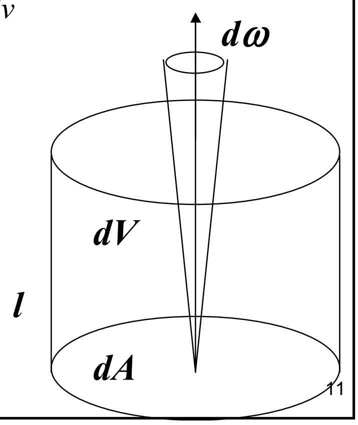
hence, the energy contained in volume element dV per frequency interval is

$$u_\nu dV \; d\nu = \oint_{4\pi} I_\nu \; d\omega \, d\nu \; dt \; dA = 4\pi J_\nu \; dV / c \; d\nu$$

$$u_\nu = \tfrac{4\pi}{c} J_\nu \qquad \frac{\text{energy}}{\text{volume frequency}} \quad \frac{\text{erg}}{\text{cm}^3 \ \text{Hz}}$$

total radiation energy in volume element:

$$\boxed{u = \int_0^\infty u_\nu \; d\nu = \tfrac{4\pi}{c}\int_0^\infty J_\nu \; d\nu} \qquad \frac{\text{energy}}{\text{volume}} \quad \frac{\text{erg}}{\text{cm}^3}$$

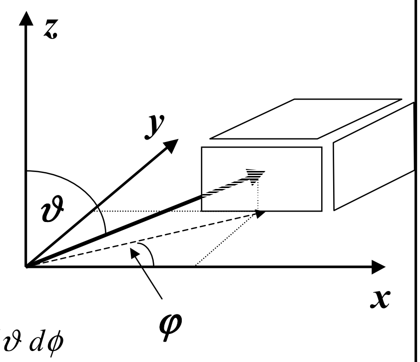


The 1st moment: radiation flux

$$\vec{F}_\nu = \oint_{4\pi} I_\nu(\vec{n}) \; \vec{n} \; d\omega$$

propagation vector in spherical coordinates:

$$\vec{n} = \begin{pmatrix} \sin\vartheta\cos\phi \\ \sin\vartheta\sin\phi \\ \cos\vartheta \end{pmatrix}$$



$$\Rightarrow F_{\nu,x} = \iint I(\vartheta,\phi)\sin\vartheta\cos\phi\sin\vartheta \; d\vartheta \; d\phi$$

in plane-parallel or spherical geometry:

$$F_{\nu,x} = F_{\nu,y} = 0, \; F_{\nu,z} = \boxed{F_\nu = 2\pi\int_{-1}^{1} I_\nu(\mu)\mu \; d\mu}$$

$$\frac{\text{energy}}{\text{area time frequency}} \quad \frac{\text{erg}}{\text{cm}^2 \ \text{s Hz}}$$

Meaning of flux:

Radiation flux = netto energy going through area ⊥ z-axis
Decomposition into two half-spaces:

$$\begin{aligned} F &= 2\pi \int_{-1}^{1} I(\mu)\mu d\mu \\ &= 2\pi \int_{0}^{1} I(\mu)\mu d\mu + 2\pi \int_{-1}^{0} I(\mu)\mu d\mu \\ &= 2\pi \int_{0}^{1} I(\mu)\mu d\mu - 2\pi \int_{0}^{1} I(-\mu)\mu d\mu \\ &= F^{+} - F^{-} \end{aligned}$$

netto = outwards - inwards

Special case: isotropic radiation field: $F = 0$

Other definitions:

$$\boxed{\begin{array}{ll} \mathrm{F}_\nu & \text{astrophysical flux} \\ H_\nu & \text{Eddington flux} \\ \multicolumn{2}{l}{F_\nu = \pi \mathrm{F}_\nu = 4\pi H_\nu} \end{array}}$$

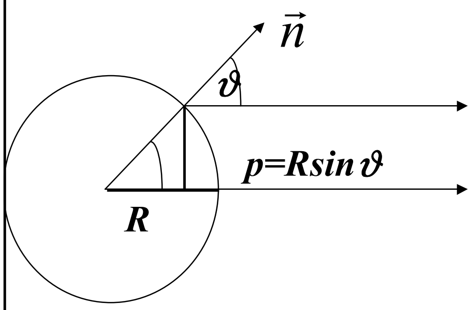
Idea behind definition of Eddington flux

In 1-dimensional geometry the n-th moments of intensity are

$$\boxed{\begin{array}{ll} \text{0-th moment:} & J_\nu = \frac{1}{2}\int_{-1}^{1} I(\mu)\, d\mu \\ \text{1st moment:} & H_\nu = \frac{1}{2}\int_{-1}^{1} I(\mu)\, \mu d\mu \\ \text{2nd moment:} & K_\nu = \frac{1}{2}\int_{-1}^{1} I(\mu)\, \mu^2 d\mu \\ \text{n-th moment:} & = \frac{1}{2}\int_{-1}^{1} I(\mu)\, \mu^{n} d\mu \end{array}}$$

Idea behind definition of astrophysical flux

Intensity averaged over stellar disk = astrophysical flux



$$p = R\sin\vartheta$$

$$p^2 = R^2(1-\mu^2)$$

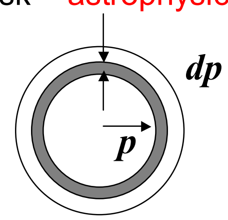
$$2p\frac{dp}{d\mu} = -2R^2\mu$$

$$pdp = -R^2\mu d\mu$$

Stellar Atmospheres: The Radiation Field

Idea behind definition of astrophysical flux

Intensity averaged over stellar disk = astrophysical flux



$$\bar{I}_\nu = \frac{1}{\pi R^2}\int_0^R I_\nu(p) 2\pi\, pdp$$

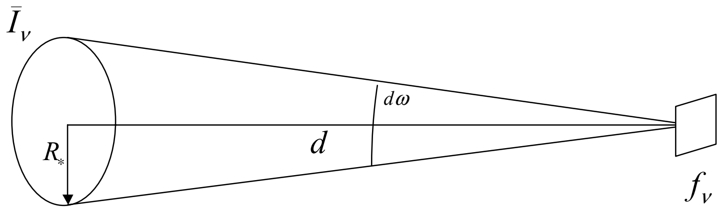
$$= \frac{1}{\pi R^2}\int_0^1 I_\nu(\mu) 2\pi R^2 \mu d\mu$$

$$= F_\nu^+ / \pi = \mathrm{F}_\nu^+$$

$\mathrm{F}_\nu^- = 0$ no inward flux at stellar surface

$$\boxed{\bar{I}_\nu = \mathrm{F}_\nu}$$

Flux at location of observer



$$\vec F_\nu = \oint_{4\pi} I_\nu(\vec n)\, \vec n\, d\omega$$

Flux at distant observer's detector normal to the line of sight:

$$f_\nu = \bar I_\nu d\omega = \bar I_\nu \pi R_*^2 / d^2 = \pi \mathrm{F}_\nu \frac{R_*^2}{d^2} = F_\nu \frac{R_*^2}{d^2}$$

Total energy radiated away by the star, luminosity

Integral over frequency at outer boundary:

$$F = \int_0^\infty F_\nu^+ d\nu = \int_0^\infty F_\nu\, d\nu$$

Multiplied by stellar surface area yields the luminosity

$$\boxed{L = 4\pi R_*^2 F = 4\pi^2 R_*^2 \mathrm{F} = 16\pi^2 R_*^2 H}$$

$$\frac{\text{energy}}{\text{time}} \quad \frac{\text{erg}}{\text{s}}$$

The photon gas pressure

Photon momentum: $p_\nu = E_\nu / c$

Force: $F = \dfrac{dp_{\nu\perp}}{dt} = \dfrac{1}{c}\dfrac{dE_\nu}{dt}\cos\vartheta$

Pressure:
$$dP_\nu = \frac{F}{dA} = \frac{1}{c}\frac{dE_\nu \cos\vartheta}{dt \quad dA}$$
$$= \frac{1}{c} I_\nu \cos^2\vartheta \, d\omega \, d\nu$$

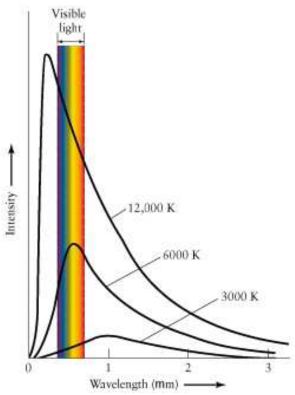
$$\boxed{P(\nu) = \frac{1}{c}\oint_{4\pi} I_\nu \cos^2\vartheta \, d\omega = \frac{2\pi}{c}\int_{-1}^{1} I_\nu \mu^2 d\mu = \frac{4\pi}{c} K_\nu}$$

Isotropic radiation field: $I_\nu(\mu) = I_\nu = J_\nu$

$$P(\nu) = \frac{4\pi}{c}\frac{I_\nu}{3} \quad u_\nu = \frac{4\pi}{c} I_\nu \Rightarrow P(\nu) = \frac{1}{3} u_\nu \quad J_\nu = 3K_\nu$$

Special case: black body radiation (Hohlraumstrahlung)

Radiation field in Thermodynamic Equilibrium with matter of temperature T



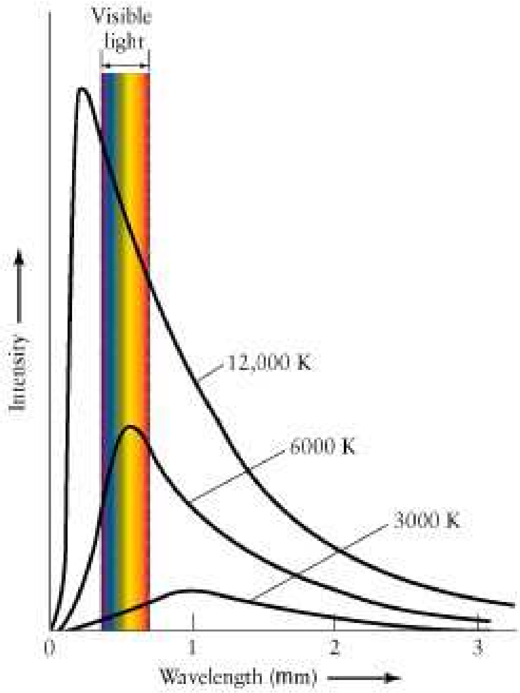
$$I_\nu(\nu, \vec{n}, \vec{r}, t) = I_\nu(\nu)$$
$$I_\nu = B_\nu(\nu, T) \text{ bzw. } I_\lambda = B_\lambda(\nu, T)$$
$$\text{in cavity: } \vec{F} = 0 \quad J_\nu = I_\nu = B_\nu$$

$$B_\nu(\nu, T) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}$$

$$B_\nu(\lambda, T) = \frac{2hc}{\lambda^3}\left[\exp\left(\frac{hc}{\lambda kT}\right) - 1\right]^{-1}$$

$$B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5}\left[\exp\left(\frac{hc}{\lambda kT}\right) - 1\right]^{-1}$$

$$B_\lambda(\nu, T) = \frac{2h\nu^5}{c^3}\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}$$



Asymptotic behaviour

- In the „red“ Rayleigh-Jeans domain

$$\frac{h\nu}{kT} << 1 \qquad \exp\left(\frac{h\nu}{kT}\right) \approx 1 + \frac{h\nu}{kT}$$

$$B_\nu(\nu,T) = \frac{2k\nu^2 T}{c^2}$$

$$\boxed{B_\nu(\lambda,T) = \frac{2ckT}{\lambda^4}}$$

- In the „blue“ Wien domain

$$\frac{h\nu}{kT} >> 1 \qquad \exp\left(\frac{h\nu}{kT}\right) - 1 \approx \exp\left(\frac{h\nu}{kT}\right)$$

$$B_\nu(\nu,T) = \frac{2h\nu^3}{c^2}\exp\left(-\frac{h\nu}{kT}\right)$$

$$\boxed{B_\nu(\lambda,T) = \frac{2hc}{\lambda^3}\exp\left(-\frac{hc}{\lambda kT}\right)}$$

Wien´s law

$$\frac{d}{d\nu}B_\nu(\nu,T) = \frac{d}{d\nu}\left[\frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}\right]$$

x:=hv/kT

$$= B_\nu\left[\frac{3}{\nu}+\frac{-1}{\mathrm{e}^x-1}\frac{x}{\nu}\mathrm{e}^x\right]$$

$$\frac{d}{d\nu}B_\nu = 0 \rightarrow 3 - x_{\max}\,\mathrm{e}^{x_{\max}}/\left(\mathrm{e}^{x_{\max}}-1\right)=0$$

$$\rightarrow x_{\max} - 3\left(1-\mathrm{e}^{-x_{\max}}\right)=0$$

numerical solution: $\boxed{x_{\max} = 2.821 = \frac{h\nu_{\max}}{kT}}$ $\lambda_{\max}T = 0.5100\,\mathrm{cm\ \ deg}$

$$\frac{d}{d\lambda}B_\lambda = 0 \rightarrow x_{\max} - 5\left(1-\mathrm{e}^{-x_{\max}}\right)=0$$

numerical solution: $\boxed{x_{\max} = 4.965 = \frac{hc}{\lambda_{\max}kT}}$ $\lambda_{\max}T = 0.2897\,\mathrm{cm\ \ deg}$

Integration over frequencies

$$B(T)=\int_0^\infty B_\nu(T)d\nu = \int_0^\infty \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}d\nu$$

$$=\frac{2k^4}{c^2h^3}T^4\underbrace{\int_0^\infty\frac{x^3}{e^x-1}dx}_{=\pi^4/15}=\frac{2}{15}\frac{\pi^4k^4}{c^2h^3}T^4$$

$$=\frac{\sigma}{\pi}T^4 \quad \text{with } \sigma=\frac{2}{15}\frac{\pi^5k^4}{c^2h^3}=5.669\cdot10^{-5}\,\mathrm{erg\ cm^{-2}\ s^{-1}\ deg^{-4}}$$

Stefan-Boltzmann law

Energy density of blackbody radiation:

$$\boxed{u=\frac{4\pi}{c}\int_0^\infty J_\nu(\nu)d\nu=\frac{4\pi}{c}B(T)=\frac{4\sigma}{c}T^4}$$

Stars as black bodies – effective temperature

Surface as „open" cavity (... physically nonsense)

$$I_\nu^+ = B_\nu \ , I_\nu^- = 0$$

$$I_\nu = \begin{cases} B_\nu & \text{for } \mu > 0 \\ 0 & \text{for } \mu \le 0 \end{cases}$$

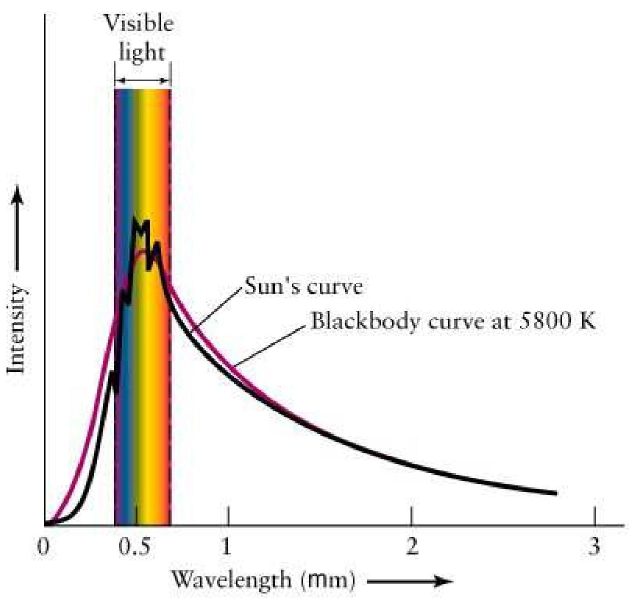
$$\text{with } \ \mathrm{F}_\nu = B_\nu \ \text{ and } \ \mathrm{F} = B(T) = \frac{\sigma}{\pi}\mathrm{T}^4$$

$$\text{luminosity:} \qquad L = 4\pi^2 R_*^2 \mathrm{F} = 4\sigma\pi R_*^2 T^4$$

$$\text{hence, eff. temperature:} \ \boxed{T_{\text{eff}} = \left(4\sigma\pi\right)^{-1/4} L^{-1/4} R_*^{-1/2}}$$

Attention: definition dependent on stellar radius!

Stars as black bodies – effective temperature



Examples and applications

- Solar constant, effective temperature of the Sun

$$\int_0^\infty f_\nu(\nu)d\nu = f = 1.36 \text{ kW/m}^2 = 1.36 \text{ erg s}^{-1} \text{ cm}^{-2}$$

$$\mathrm{F} = f\frac{d^2}{\pi R_*^2} \quad \text{with} \quad d = 1.5\cdot 10^{13}\,\text{cm} \quad R_\odot = 6.69\cdot 10^{10}\,\text{cm}$$

$$\mathrm{F}_\odot = 2.01\cdot 10^{10} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ flux at solar surface}$$

$$T_\text{eff}^4 = \frac{\pi}{\sigma}\mathrm{F} \Rightarrow T_\text{eff}^\odot = 5780 \text{ K}$$

- Sun‘s center

$$T_c = 1.4\cdot 10^7 \text{ K}$$

$$\Rightarrow \text{Planck maximum at} \quad \lambda_\text{max} = 3.4\text{Å} \quad (B_\nu)$$

$$\text{or} \quad \lambda_\text{max} = 2.1\text{Å} \quad (B_\lambda)$$

$$\text{with} \quad 1\text{Å} \,\hat{=}\, 12.4 \text{ keV} \quad \text{maximum} \approx 4 \text{ keV}$$

Stellar Atmospheres: The Radiation Field

Examples and applications

- Main sequence star, spectral type O

$$R_* = 10 R_\odot \ , \ T_\text{eff}^* = 60000 \text{ K}$$

$$\frac{L_*}{L_\odot} = \left(\frac{T_\text{eff}^*}{T_\text{eff}^\odot}\right)^4 \left(\frac{R_*}{R_\odot}\right)^2 \Rightarrow L_* = 10^6 L_\odot$$

$$\lambda_\text{max} = 882\text{Å} \ (B_\nu) \quad \text{or} \quad \lambda_\text{max} = 501\text{Å} \ (B_\lambda)$$

- Interstellar dust

$$T = 20\,\text{K} \ , \lambda_\text{max} = 0.3\,\text{mm}\,(B_\nu)$$

- 3K background radiation

$$T = 2.7\,\text{K} \ , \lambda_\text{max} = 1.9\,\text{mm}\,(B_\nu)$$

Radiation temperature

... is the temperature, at which the corresponding blackbody would have equal intensity

$$I_\nu(\lambda) = \frac{2hc}{\lambda^3}\left[\exp\left(\frac{hc}{\lambda k T_{\mathrm{rad}}}\right) - 1\right]^{-1} \Rightarrow T_{\mathrm{rad}} = \frac{hc}{k\lambda}\left[\ln\left(\frac{2hc}{\lambda^3 I_\nu} + 1\right)\right]^{-1}$$

Comfortable quantity with Kelvin as unit

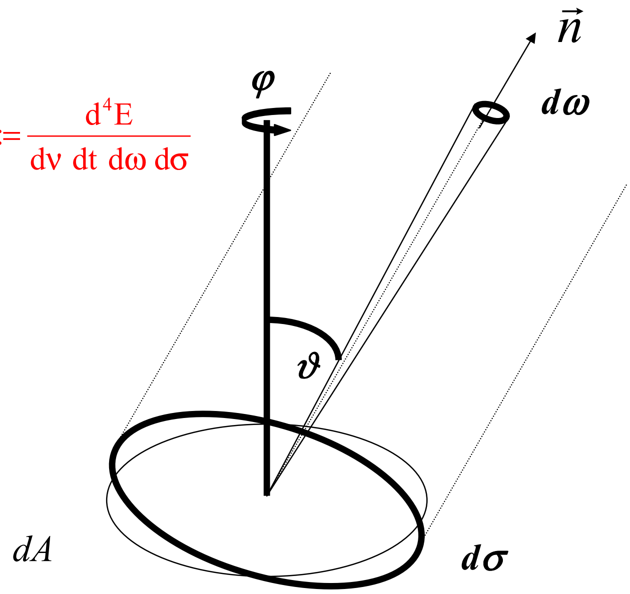
Often used in radio astronomy

The Radiation Field

- Summary -

Summary: Definition of specific intensity

$$I_\nu(\nu,\vec{n},\vec{r},t) := \frac{d^4E}{d\nu\ dt\ d\omega\ d\sigma}$$



Summary: Moments of radiation field

In 1-dim geometry (plane-parallel or spherically symmetric):

0-th moment:	$J_\nu = \frac{1}{2}\int_{-1}^{1} I(\mu)\, d\mu$	Mean intensity
1st moment:	$H_\nu = \frac{1}{2}\int_{-1}^{1} I(\mu)\, \mu d\mu$	Eddington flux
2nd moment:	$K_\nu = \frac{1}{2}\int_{-1}^{1} I(\mu)\, \mu^2 d\mu$	K-integral

F_ν =astrophysical flux H_ν=Eddington flux F_ν = flux $F_\nu = \pi \mathrm{F}_\nu = 4\pi H_\nu$

energy density $u = \int_0^\infty u_\nu\ d\nu = \frac{4\pi}{c}\int_0^\infty J_\nu\ d\nu$

total flux at stellar surface $F = \int_0^\infty F_\nu^+ d\nu = \int_0^\infty F_\nu\, d\nu$

stellar luminosity $L = 4\pi R_*^2 F = 4\pi^2 R_*^2 \mathrm{F} = 16\pi^2 R_*^2 H$

Summary: Moments of radiation field

pressure of photon gas $\mathrm{P}(\nu) = \frac{4\pi}{\mathrm{c}} \mathrm{K}_\nu$

blackbody radiation $B_\nu(\nu, T) = \frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}$

Wien´s law $\lambda_{\max} T = \text{constant}$

Stefan-Boltzmann law $B(T) = \int_0^\infty B_\nu(T)\,d\nu = \frac{\sigma}{\pi} T^4$

energy density of blackbody radiation $\mathrm{u} = \frac{4\sigma}{\mathrm{c}} \mathrm{T}^4$

effective temperature $L = 4\pi^2 R_*^2 \mathrm{F} = 4\pi^2 R_*^2 B = 4\sigma\pi R_*^2 T_{\text{eff}}^4$

Radiation Transfer

Interaction radiation – matter

Energy can be removed from, or delivered to, the radiation field

Classification by physical processes:

True absorption:	photon is destroyed, energy is transferred into kinetic energy of gas; photon is thermalized
True emission:	photon is generated, extracts kinetic energy from the gas
Scattering:	photon interacts with scatterer → direction changed, energy slightly changed → no energy exchange with gas

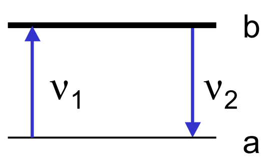
Examples: true absorption and emission

- photoionization (bound-free) excess energy is transferred into kinetic energy of the released electron $\rightarrow$ effect on local temperature
- photoexcitation (bound-bound) followed by electron collisional de-excitation; excitation energy is transferred to the electron $\rightarrow$ effect on local temperature
- photoexcitation (bound-bound) followed by collisional ionization
- reverse processes are examples for true emission

Stellar Atmospheres: Radiation Transfer

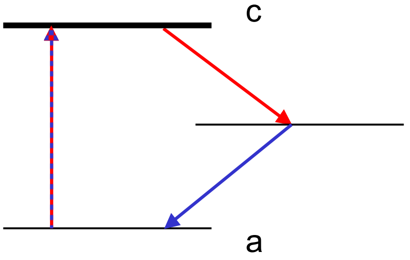
Examples: scattering processes

- 2-level atom absorbs photon with frequency ν_1, re-emits photon with frequency ν_2; frequencies not exactly equal, because
  - levels a and b have non-vanishing energy width
  - Doppler effect because atom moves
- Scattering of photons by free electrons: Compton- or Thomson scattering, (anelastic or elastic) collision of a photon with a free electron



Fluorescence

Neither scattering nor true absorption process



c-b: collisional de-excitation

b-a: radiative

Change of intensity along path element

generally: $\frac{dI_\nu}{ds}$

plane-parallel geometry: $\frac{dI_\nu}{ds} = -\mu \frac{dI_\nu}{dt}$ with $dt = -\mu ds$

spherical geometry:

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \mu}\frac{d\mu}{ds}$$

$$\Rightarrow \frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\mu + \frac{\partial I_\nu}{\partial \mu}\frac{1-\mu^2}{r}$$

$$dr = ds\cos\vartheta \Rightarrow \frac{dr}{ds} = \mu$$

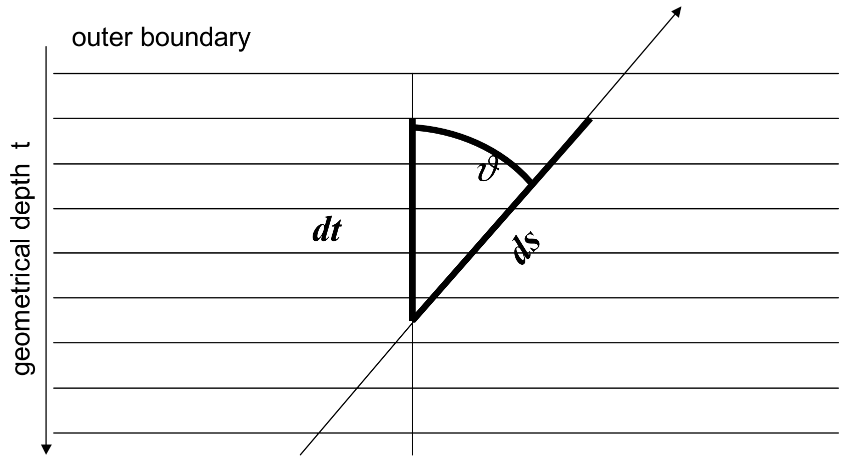
$$\sin(\vartheta + d\vartheta) \approx \sin\vartheta = \frac{-r\,d\vartheta}{ds}$$

$$\Rightarrow \frac{d\mu}{ds} = \frac{d\mu}{d\vartheta}\frac{d\vartheta}{ds} = -\sin\vartheta \frac{1}{r}(-\sin\vartheta)$$

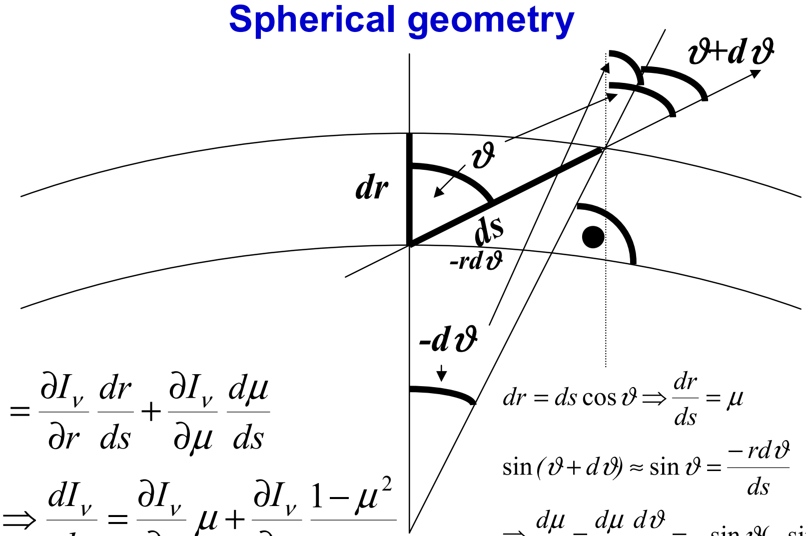
$$= (1-\mu^2)/r$$

Plane-parallel geometry

$$\frac{dI_\nu}{ds} = -\mu \frac{dI_\nu}{dt} \quad \text{with } dt = -\mu ds$$



Spherical geometry



$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \mu}\frac{d\mu}{ds}$$

$$\Rightarrow \frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\mu + \frac{\partial I_\nu}{\partial \mu}\frac{1-\mu^2}{r}$$

$$dr = ds\cos\vartheta \Rightarrow \frac{dr}{ds} = \mu$$

$$\sin(\vartheta + d\vartheta) \approx \sin\vartheta = \frac{-rd\vartheta}{ds}$$

$$\Rightarrow \frac{d\mu}{ds} = \frac{d\mu}{d\vartheta}\frac{d\vartheta}{ds} = -\sin\vartheta(-\sin\vartheta\frac{1}{r})$$

$$= (1-\mu^2)/r$$

Change of intensity along path element

generally: $\frac{dI_\nu}{ds}$

plane-parallel geometry: $\boxed{\frac{dI_\nu}{ds} = -\mu \frac{dI_\nu}{dt}}$ with $dt = -\mu ds$

spherical geometry:

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \mu}\frac{d\mu}{ds}$$

$$\Rightarrow \boxed{\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\mu + \frac{\partial I_\nu}{\partial \mu}\frac{1-\mu^2}{r}}$$

$$dr = ds\cos\vartheta \Rightarrow \frac{dr}{ds} = \mu$$

$$\sin(\vartheta + d\vartheta) \approx \sin\vartheta = \frac{-r\,d\vartheta}{ds}$$

$$\Rightarrow \frac{d\mu}{ds} = \frac{d\mu}{d\vartheta}\frac{d\vartheta}{ds} = -\sin\vartheta\frac{1}{r}(-\sin\vartheta)$$

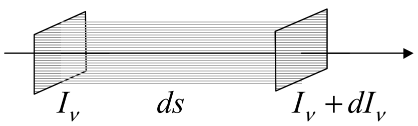
$$= (1-\mu^2)/r$$

Right-hand side of transfer equation

- No absorption (vacuum)

  $\frac{dI_\nu}{ds} = 0 \Rightarrow I_\nu = \text{const.}$ invariance of intensity

- Absorption only, no emission



energy removed from ray: $dE = -dI_\nu\, d\nu\, dt\, d\omega\, d\sigma$

is proportional to energy content in ray: $I_\nu\, d\nu\, dt\, d\omega\, d\sigma$

and to the path element: ds

Absorption coefficient

- thus: $dI_\nu = -\kappa I_\nu \, ds$
- κ absorption coefficient, opacity
- dimension: 1/length unit: cm^{-1}
- but also often used: mass absorption coefficient, e.g., per gram matter
- κ in general complicated function of physical quantities T, P, and frequency, direction, time...

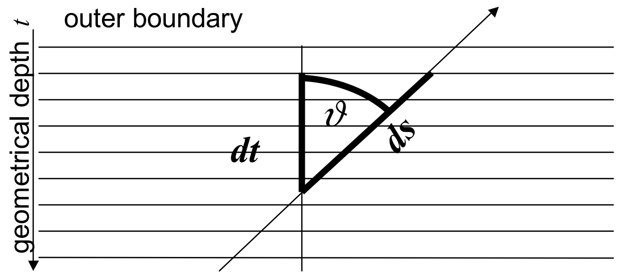
$$\kappa = \kappa(\vec{r}, \vec{n}, \nu, t)$$

- often there is a coordinate system in which κ isotropic, e.g. co-moving frame in moving atmospheres

$$\kappa = \kappa(\vec{r}, \nu)$$

- counter-example: magnetic fields (Zeeman effect)

only absorption, plane-parallel geometry



$$\frac{dI_\nu(\mu,t)}{ds} = -\kappa(\nu,t) I_\nu(\mu,t) \Rightarrow -\mu \frac{dI_\nu}{dt}(\mu,t) = -\kappa(\nu,t) I_\nu(\mu,t)$$

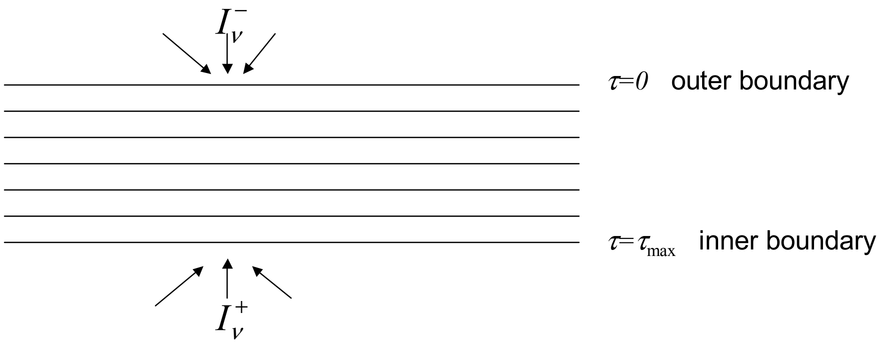
with optical depth $d\tau := \kappa dt \rightarrow \tau(\nu,t) = \int_{t=0}^{t} \kappa(\nu,t')dt'$ with $\tau = 0$ at $t = 0$

$$\Rightarrow \frac{dI_\nu(\mu,\tau)}{d\tau} = \frac{1}{\mu} I_\nu(\mu,\tau)$$

$$\Rightarrow \boxed{I_\nu(\mu,\tau) = c \cdot e^{\tau/\mu}}$$

c integration constant, fixed by boundary values

Schuster boundary-value problem



$$\mu < 0 : I_\nu^-(\mu, \tau = 0) = c \cdot e^{0/\mu} = c$$

$$\boxed{I_\nu^-(\mu, \tau) = I_\nu^-(\mu, \tau = 0) e^{-\tau/|\mu|}}$$

$$\mu > 0 : I_\nu^+(\mu, \tau = \tau_{\max}) = c \cdot e^{\tau_{\max}/\mu}$$

$$\boxed{I_\nu^+(\mu, \tau) = I_\nu^+(\mu, \tau = \tau_{\max}) e^{-(\tau_{\max} - \tau)/\mu}}$$

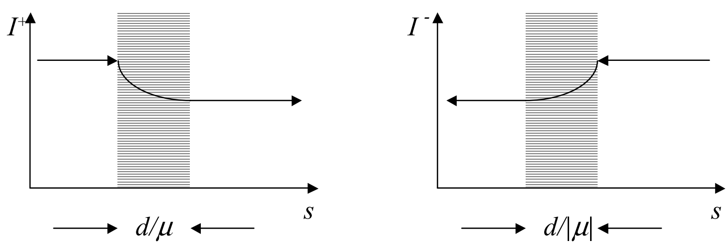
Example: homogeneous medium

e.g. glass filter

$$\kappa(t, \mu) = \kappa \rightarrow \tau = \kappa\, t \rightarrow \tau_{\max} = \kappa\, d \qquad d = \text{thickness of filter}$$

$$I_\nu^+(\mu, \tau = 0) \qquad = I_\nu^+(\mu, \tau = \tau_{\max}) \cdot e^{-(\kappa\, d - 0)/\mu}$$

$$I_\nu^-(\mu, \tau = \tau_{\max}) = I_\nu^-(\mu, \tau = 0) \cdot \qquad e^{-\kappa\, d/|\mu|}$$



Half-width thickness

$$s_{1/2}: \quad e^{-\kappa\, s_{1/2}} = 1/2$$

Material	$S_{1/2}$ / meter
River water	0.033
Window glass	0.066
City air	330
Glas fiber	6600
Solar atmosphere	200000

Physical interpretation of optical depth

What is the mean penetration depth of photons into medium?

$$\langle \tau \rangle = \int_0^\infty \tau\, p(\tau)\, d\tau \qquad \text{(mathematically: expectation value of probability function } p(\tau)\text{)}$$

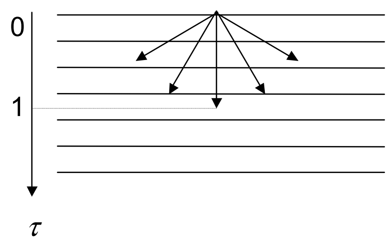
$$p(\tau)\, d\tau := \text{ probability for absorption in interval } [\tau, \tau + d\tau]$$

$$= \underbrace{e^{-\tau/\mu}}_{I_\nu^-(\tau)/I_\nu^-(\tau=0)} \cdot \frac{1}{\mu} d\tau \quad \text{note normalization: } \int_0^\infty p(\tau) d\tau = \int_0^\infty e^{-\tau/\mu} \frac{d\tau}{\mu} = -e^{-x}\Big|_0^\infty = 1$$

$$\langle \tau \rangle = \int_0^\infty \tau\, \mathrm{e}^{-\tau/\mu} \cdot \frac{1}{\mu} d\tau = \mu \int_0^\infty x e^{-x} dx = \mu \cdot 1$$

$$\langle \tau \rangle = \mu \quad \text{mean penetration depth}$$

$$\mathrm{s_f} = \frac{t_f}{\mu} = \frac{1}{\mu} \frac{\langle \tau \rangle}{\kappa} = \boxed{\frac{1}{\kappa} \quad \text{mean free path}}$$



The right-hand side of the transfer equation

- transfer equation including emission

Energy added to the ray: $dE = +dI_\nu \, d\nu \, dt \, d\omega \, d\sigma$

is proportional to path element: ds

emission coefficient η_ν $\boxed{dI_\nu = \eta_\nu ds}$

- dimension: intensity / length unit: erg cm^{-3} sterad $^{-1}$

The right-hand side of the transfer equation

- Transfer equation including emission

η in general a complicated function of physical quantities T,P,..., and frequency $\eta = \eta(\vec{r}, \vec{n}, \nu, t)$

η is not isotropic even in static atmospheres, but is usually assumed to isotropic (complete redistribution)

if constant with time: $\eta = \eta(\vec{r}, \nu)$

The complete transfer equation

$$\frac{dI_\nu}{ds} = \eta_\nu - \kappa(\nu) I_\nu$$

Definition of source function: $S_\nu = \dfrac{\eta_\nu}{\kappa(\nu)}$

$$\boxed{\frac{dI_\nu}{ds} = \kappa(\nu)\left(S_\nu - I_\nu\right)}$$

- Plane-parallel geometry

$$-\mu \frac{dI_\nu(\nu,\mu,t)}{dt} = \kappa(\nu,t)\left(S_\nu(\nu,\mu,t) - I_\nu(\nu,\mu,t)\right)$$

- Spherical geometry

$$\mu \frac{\partial I_\nu(\nu,\mu,r)}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial I_\nu(\nu,\mu,r)}{\partial \mu} = \kappa(\nu,r)\left(S_\nu(\nu,\mu,r) - I_\nu(\nu,\mu,r)\right)$$

Solution with given source function: Formal solution

- Plane-parallel case

$$\frac{dI_\nu}{d\tau} = \frac{1}{\mu}\left(I_\nu - S_\nu\right) \qquad \text{or:} \qquad \frac{dI_\nu}{d\tau} + \frac{1}{-\mu} I_\nu = \frac{1}{-\mu} S_\nu$$

linear 1st-order differential equation of form $y' + f(x) y = g(x)$

has the integrating factor $M(x) = \exp\left(\int_{x_0}^{x} f(x)dx\right)$

und thus the solution $y(x) = \dfrac{1}{M(x)}\left(\int_{x_0}^{x} g(x)M(x)dx + C\right)$ $C = y(x_0)$

(proof by insertion)

in our case:

$$x \to \tau_\nu$$
$$f(x) \to -1/\mu$$
$$g(x) \to -1/\mu\, S_\nu(\tau_\nu)$$
$$y(x) \to I_\nu(\tau_\nu)$$

Formal solution for I^+

Reference point x_0: $\tau = \tau_{\max}$ for I^+ ($\mu > 0$) outgoing radiation

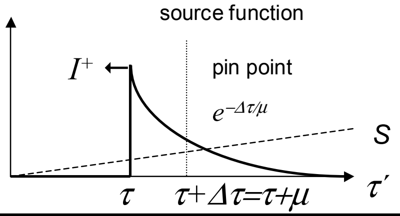
$$M(\tau) = \exp\left(\int_{\tau_{\max}}^{\tau} -\frac{1}{\mu}\, d\tau'\right) = \exp\left(\frac{\tau_{\max}-\tau}{\mu}\right)$$

$$I_\nu^+(\tau) = \exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right)\left[\int_{\tau_{\max}}^{\tau} -\frac{1}{\mu} S_\nu(\tau') \exp\left(\frac{\tau_{\max}-\tau'}{\mu}\right) d\tau' + I_\nu^+(\tau_{\max})\right]$$

$$\boxed{I_\nu^+(\tau) = \int_{\tau}^{\tau_{\max}} S_\nu(\tau') \exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu} + I_\nu^+(\tau_{\max}) \exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right)}$$

weighted mean over source function

exponentially absorbed ingoing radiation from inner boundary



Hence, as rough approximation:

$$I_\nu^+(\tau) \approx S_\nu(\tau+\mu)$$

Formal solution for I^-

Reference point x_0: $\tau=0$ for I^- ($\mu < 0$) ingoing radiation

$$M(\tau) = \exp\left(\int_{0}^{\tau} -\frac{1}{\mu}\, d\tau'\right) = \exp\left(\frac{\tau}{|\mu|}\right)$$

$$I_\nu^-(\tau) = \exp\left(-\frac{\tau}{|\mu|}\right)\left[\int_{0}^{\tau} -\frac{1}{\mu} S_\nu(\tau') \exp\left(\frac{\tau'}{|\mu|}\right) d\tau' + I_\nu^-(0)\right]$$

$$\boxed{I_\nu^-(\tau) = \int_{0}^{\tau} S_\nu(\tau') \exp\left(-\frac{\tau-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|} + I_\nu^-(0) \exp\left(-\frac{\tau}{|\mu|}\right)}$$

weighted mean over source function

exponentially absorbed ingoing radiation from outer boundary

Emergent intensity

$$I_\nu^+(0) = \int_0^{\tau_{\max}} S_\nu(\tau')\exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu} + I_\nu^+(\tau_{\max})\exp\left(-\frac{\tau_{\max}}{\mu}\right)$$

for semi-infinite atmospheres: $\tau_{\max} \to \infty$:

$$I_\nu^+(0) = \int_0^{\tau_{\max}} S_\nu(\tau')\exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu}$$

hence, approximately: $I_\nu^+(0) \approx S_\nu(\tau = \mu)$

Eddington-Barbier-Relation

Relation is exactly valid if source function is linear in τ:

i.e. with $S_\nu(\tau) = S_{0\nu} + S_{1\nu}\cdot\tau$ and $x := \tau'/\mu$ we have:

$$I_\nu^+(0) = S_{0\nu}\int_0^\infty e^{-x}dx + S_{1\nu}\int_0^\infty \mu\, xe^{-x}dx = S_{0\nu} + S_{1\nu}\cdot\mu = S_\nu(\mu)$$

The source function

In thermodynamic equilibrium (TE): for any volume element it is:

absorbed energy = emitted energy

per second per second

$$\kappa I_\nu\, ds d\sigma d\omega d\nu = \eta_\nu\, ds d\sigma d\omega d\nu$$

$$\kappa B_\nu = \eta_\nu$$ Kirchhoff´s law

$$S_\nu = \frac{\eta_\nu}{\kappa} = B_\nu$$

The local thermodynamic equilibrium (**LTE**): we assume that

$$S_\nu(\nu,\vec r) = B_\nu(\nu, T(\vec r)) \qquad \text{z.B. } I_\nu^+(0) = \int_0^{\tau_{\max}} B_\nu(T(\tau'))\exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu}$$

Local temperature, unfortunately unknown at the outset

In stellar atmospheres TE is not fulfilled, because

– System is open for radiation
– $T(r)\neq$const (temperature gradient)

Source function with scattering

Example: thermal absorption + continuum scattering

(Thomson scattering of free electrons)

$$\kappa(\nu) = \chi(\nu) + \sigma(\nu) \qquad \eta_\nu = \chi B_\nu + \sigma \oint \frac{d\omega}{4\pi} \int_0^\infty R(\nu', \vec{n}'; \nu, \vec{n}) I_\nu(\nu', \vec{n}') d\nu'$$

true absorption

scattering

redistribution function

isotropic, coherent: $R(\nu', \vec{n}'; \nu, \vec{n}) = \delta(\nu', \nu)$

$$\eta_\nu = \chi B_\nu + \sigma J_\nu$$

$$S_\nu = \frac{\chi B_\nu + \sigma J_\nu}{\chi + \sigma} = \rho J_\nu + (1-\rho) B_\nu \quad \text{with} \quad \rho = \sigma / (\sigma + \chi)$$

Inserting into formal solution:

$$I_\nu^+(0) = \int_0^\infty (1-\rho) B_\nu \exp\left(-\frac{\tau'}{\mu}\right) \frac{d\tau'}{\mu} + \int_0^\infty \rho J_\nu \exp\left(-\frac{\tau'}{\mu}\right) \frac{d\tau'}{\mu}$$

$$\oint \frac{d\omega}{4\pi} I_\nu(\tau, \mu)$$ integral equation for I_ν

Stellar Atmospheres: Radiation Transfer

The Schwarzschild-Milne equations

Expressions for moments of radiation field obtained by integration of formal solution over angles μ

0-th moment

$$J_\nu(\tau) = \frac{1}{2} \int_{-1}^{1} I_\nu(\tau, \mu) d\mu$$

$$J_\nu(\tau) = \frac{1}{2}\left[\int_0^1 d\mu \int_\tau^\infty S_\nu(\tau') \exp\left(-\frac{\tau'-\tau}{\mu}\right) \frac{d\tau'}{\mu} + \int_{-1}^{0} d\mu \int_0^\tau S_\nu(\tau') \exp\left(-\frac{\tau-\tau'}{|\mu|}\right) \frac{d\tau'}{|\mu|} \right]$$

(written for semi-infinite atmosphere without irradiation from outside)

exchange integrals (S, τ independent of μ) $w = \frac{1}{|\mu|}, \frac{dw}{d\mu} = \mp \frac{1}{\mu^2} \rightarrow d\mu = \mp \frac{dw}{w^2}$

$$J_\nu(\tau) = \frac{1}{2}\left[\int_\tau^\infty d\tau' S_\nu(\tau') \int_\infty^1 \exp\left(-w(\tau'-\tau)\right) w \left(-\frac{dw}{w^2}\right) + \int_0^\tau d\tau' S_\nu(\tau') \int_1^\infty \exp\left(-w(\tau-\tau')\right) w \left(\frac{dw}{w^2}\right) \right]$$

$$J_\nu(\tau) = \frac{1}{2}\left[\int_\tau^\infty d\tau' S_\nu(\tau') \int_1^\infty \exp\left(-w(\tau'-\tau)\right) \left(\frac{dw}{w}\right) + \int_0^\tau d\tau' S_\nu(\tau') \int_1^\infty \exp\left(-w(\tau-\tau')\right) \left(\frac{dw}{w}\right) \right]$$

The Schwarzschild-Milne equations

0-th moment

$$J_\nu(\tau)=\frac{1}{2}\left[\int_\tau^\infty d\tau' S_\nu(\tau')\int_1^\infty \exp\left(-w(\tau'-\tau)\right)\left(\frac{dw}{w}\right)+\int_0^\tau d\tau' S_\nu(\tau')\int_1^\infty \exp\left(-w(\tau-\tau')\right)\left(\frac{dw}{w}\right)\right]$$

$$J_\nu(\tau)=\frac{1}{2}\left[\int_\tau^\infty S_\nu(\tau')E_1\left(\tau'-\tau\right)d\tau'+\int_0^\tau S_\nu(\tau')E_1\left(\tau-\tau'\right)d\tau'\right]$$

with $E_1(\mathrm{x}):=\int_1^\infty \mathrm{t}^{-1}e^{-xt}dt$ exponential integral of 1st order

$$\boxed{J_\nu(\tau)=\frac{1}{2}\int_0^\infty S_\nu(\tau')E_1\left(\left|\tau'-\tau\right|\right)d\tau'}$$

Karl Schwarzschild (1914)

The Lambda operator

Definition $\Lambda\left[f(t)\right]=\frac{1}{2}\int_0^\infty f(t)E_1\left(\left|t-\tau\right|\right)dt$

$$\Rightarrow \boxed{J_\nu(\tau)=\Lambda\left(S_\nu\right)}$$

In analogy, we obtain the Milne equations for the

1st moment

$$H_\nu(\tau)=\frac{1}{2}\int_\tau^\infty S_\nu(t)E_2\left(t-\tau\right)dt-\frac{1}{2}\int_0^\tau S_\nu(t)E_2\left(\tau-t\right)dt=\frac{1}{4}\Phi\left(S_\nu\right)$$

2nd moment

$$K_\nu(\tau)=\frac{1}{2}\int_0^\infty S_\nu(t)E_3\left(\left|t-\tau\right|\right)dt=\frac{1}{4}\mathrm{X}\left(S_\nu\right)$$

with $E_n(x)=\int_1^\infty \frac{e^{-xt}}{t^n}dt$

LTE

Strict LTE $J_\nu(\tau) = \Lambda B_\nu(T(\tau))$

Including scattering $S_\nu = \rho J_\nu + (1-\rho) B_\nu(T(\tau))$

$$J_\nu(\tau) = \Lambda \rho J_\nu + \Lambda (1-\rho) B_\nu(T(\tau))$$

Integral equation for $J_\nu(\tau)$

Solve $J_\nu(\tau) \to S_\nu(\tau) \to I_\nu(\tau)$

$$\to H_\nu(\tau) = 1/4\, \Phi S_\nu(\tau)$$

$$\to K_\nu(\tau) = 1/4\, \mathrm{X} S_\nu(\tau)$$

Excursion: exponential integral function ☛

see Chandrasekhar: Radiative Transfer III.18

- For classical LTE atmosphere models, >50% of computation time is needed to calculate $E_n(x)$
- In non-LTE models, $E_n(x)$ is needed to calculate electron collisional rates
- Recursion formula

integration by parts $E_n(x) = \int_1^\infty t^{-n} e^{-xt}\, dt$

with product rule $E_n(x) = -\frac{1}{n-1} t^{-(n-1)} e^{-xt} \Big|_1^\infty - \int_1^\infty -\frac{1}{n-1} t^{-(n-1)} (-x) e^{-xt}\, dt$

$$= 0 + \frac{1^{-(n-1)}}{n-1} e^{-x} - \frac{1}{n-1} x \int_1^\infty t^{-(n-1)} e^{-xt}\, dt$$

$$\boxed{E_n(x) = \frac{1}{n-1}\left[e^{-x} - x E_{n-1}(x) \right] \quad \text{for } n > 1}$$

$$E_1(x) \to E_n(x)$$

Excursion: exponential integral function

- differentiation

$$\frac{d}{dx}E_n(x)=\int_1^\infty t^{-n}\frac{d}{dx}(e^{-xt})dt=\int_1^\infty t^{-n}(-t)e^{-xt}dt=-\int_1^\infty t^{-(n-1)}e^{-xt}dt$$

$$\boxed{\frac{d}{dx}E_n(x)=-E_{n-1}(x)}\qquad n>1$$

$$\frac{d}{dx}E_1(x)=\int_1^\infty t^{-1}\frac{d}{dx}(e^{-xt})dt=\int_1^\infty t^{-1}(-t)e^{-xt}dt=\frac{1}{x}e^{-xt}\Big|_1^\infty=-\frac{e^{-x}}{x}$$

$$\boxed{\frac{d}{dx}E_1(x)=-\frac{e^{-x}}{x}}$$

Stellar Atmospheres: Radiation Transfer

Excursion: exponential integral function

- integrals $\int_0^s x^l E_n(x)dx$ repeated integration by parts

$$\int_0^s x^l E_n(x)dx=\frac{x^{l+1}}{l+1}E_n(x)\Big|_0^s-\int_0^s\frac{x^{l+1}}{l+1}E_n'(x)dx$$

$$=\frac{x^{l+1}}{l+1}E_n(x)\Big|_0^s+\int_0^s\frac{x^{l+1}}{l+1}E_{n-1}(x)dx \quad \text{etc. until } \frac{\mathrm{d}}{\mathrm{dx}}E_1(x)=-\frac{e^{-x}}{x}$$

$$=\frac{s^{l+1}}{l+1}E_n(s)+\frac{s^{l+2}}{(l+1)(l+2)}E_{n-1}(s)+\cdots+\frac{x^{l+n}}{(l+1)(l+2)\cdots(l+n)}E_1(s)$$

$$+\frac{1}{(l+1)(l+2)\cdots(l+n)}\int_0^s x^{l+n-1}e^{-x}dx$$

for $s\to\infty$

$$\int_0^\infty x^l E_n(x)dx=\frac{1}{(l+1)(l+2)\cdots(l+n)}\int_0^\infty x^{l+n-1}e^{-x}dx=\frac{(l+n-1)!}{(l+1)(l+2)\cdots(l+n)}=\frac{(l+n-1)!\,l!}{(l+n)!}=\frac{l!}{l+n}$$

Excursion: exponential integral function

- asymptotic behaviour

$$x \to \infty: E_1(x) = \int_1^\infty e^{-xt}\frac{1}{t}dt = \frac{e^{-x}}{x} + \int_1^\infty \frac{e^{-xt}}{x}\frac{1}{t^2}dt = \cdots = \frac{e^{-x}}{x}\left[1 - \frac{1}{x} + \frac{2}{x^2} - \frac{6}{x^3} + \cdots\right]$$

$$x \to 0: E_1(x) = \int_1^\infty e^{-xt}\frac{1}{t}dt = \int_x^\infty e^{-u}\frac{du}{u} = \int_1^\infty e^{-u}\frac{du}{u} + \int_x^1 e^{-u}\frac{du}{u}$$

$$= \int_1^\infty e^{-u}\frac{du}{u} - \int_0^1 (1-e^{-u})\frac{du}{u} + \int_x^1 \frac{du}{u} + \int_0^x (1-e^{-u})\frac{du}{u}$$

$$E_1(x) = \quad -\gamma \qquad\qquad -\ln x + \int_0^x (1-e^{-u})\frac{du}{u}$$

$\gamma = 0.5772156\cdots$ Euler´s constant

series expansion for the integral:

$$E_1(x) = -\gamma - \ln x + \sum_{n=1}^\infty (-1)^{n-1}\frac{x^n}{n!n}$$

Values at $x = 0$: $E_n(0) = \frac{1}{n-1}\left[e^{-0} - 0 \cdot E_{n-1}(0)\right] = \frac{1}{n-1} \quad n > 1 \quad E_2(0) = 1, E_3(0) = \frac{1}{2}$

Example: linear source function

$$\mathrm{S}(\tau) = \mathrm{a} + \mathrm{b}\tau$$

$$\mathrm{J}(\tau) = \Lambda\mathrm{S} = \frac{1}{2}\int_0^\infty (\mathrm{a} + \mathrm{b}\tau')\mathrm{E}_1\left(|\tau' - \tau|\right)d\tau'$$

$$= \frac{1}{2}\mathrm{a}\int_0^\infty \mathrm{E}_1\left(|\tau' - \tau|\right)d\tau' + \frac{1}{2}\mathrm{b}\int_0^\infty \tau'\mathrm{E}_1\left(|\tau' - \tau|\right)d\tau'$$

…. $$\mathrm{J}(\tau) = \mathrm{a} + \mathrm{b}\tau + \frac{1}{2}\left[\mathrm{bE}_3(\tau) - \mathrm{aE}_2(\tau)\right]$$

$$\mathrm{H}(\tau) = \frac{1}{3}\mathrm{b} + \frac{1}{2}\left[\mathrm{aE}_3(\tau) - \mathrm{bE}_4(\tau)\right]$$ ... one can show this

Conclusions: $\tau >> 1: E_n \approx e^{-x}/x \to 0 \quad J_\nu \to a + b\tau = S_\nu$

The mean intensity approaches the local source function

$$H_\nu \to b/3$$

The flux only depends on the gradient of the source function

Moments of transfer equation

- Plane-parallel geometry

$$\mu\frac{dI_\nu}{d\tau}=I_\nu-S_\nu$$

- 0-th moment

$$\frac{1}{2}\int_{-1}^{1}\cdots d\mu$$

$$\frac{d}{d\tau}\left[\frac{1}{2}\int_{-1}^{1}I_\nu(\mu)\mu d\mu\right]=\frac{1}{2}\int_{-1}^{1}I_\nu(\mu)d\mu-\frac{1}{2}\int_{-1}^{1}S_\nu d\mu$$

$$\boxed{\frac{d}{d\tau}H_\nu=J_\nu-S_\nu}\quad\text{(I)}$$

- 1st moment

$$\frac{1}{2}\int_{-1}^{1}\cdots\mu d\mu$$

$$\frac{d}{d\tau}\left[\frac{1}{2}\int_{-1}^{1}I_\nu(\mu)\mu^2 d\mu\right]=\frac{1}{2}\int_{-1}^{1}I_\nu(\mu)\mu d\mu-\frac{1}{2}\int_{-1}^{1}S_\nu\mu d\mu$$

$$\boxed{\frac{d}{d\tau}K_\nu=H_\nu}\quad\text{(II)}$$

Moments of transfer equation

- Spherical geometry

$$\mu\frac{\partial I_\nu(\nu,\mu,r)}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial I_\nu(\nu,\mu,r)}{\partial\mu}=\kappa(\nu,r)\left(S_\nu(\nu,\mu,r)-I_\nu(\nu,\mu,r)\right)$$

- 0-th moment

$$\frac{1}{2}\int_{-1}^{1}\cdots d\mu$$

$$\frac{\partial}{\partial r}\frac{1}{2}\int_{-1}^{1}I_\nu\mu d\mu+\frac{1}{2}\int_{-1}^{1}\frac{1-\mu^2}{r}\frac{\partial I_\nu}{\partial\mu}d\mu=\frac{1}{2}\kappa S_\nu\int_{-1}^{1}d\mu-\frac{1}{2}\kappa\int_{-1}^{1}I_\nu d\mu$$

$$\frac{\partial}{\partial r}H_\nu+\frac{1}{2}\left[\frac{1-\mu^2}{r}I_\nu\right]_{-1}^{1}-\frac{1}{2}\int_{-1}^{1}\frac{-2\mu}{r}I_\nu d\mu=\kappa S_\nu-\kappa J_\nu$$

$$\frac{\partial}{\partial r}H_\nu+0+\frac{2}{r}H_\nu=\kappa(S_\nu-J_\nu)$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2H_\nu\right)=\kappa(S_\nu-J_\nu)}\quad\text{(I)}$$

Moments of transfer equation

$\frac{1}{2}\int_{-1}^{1} \cdots \mu d\mu$

- 1st moment

$$\frac{\partial}{\partial r}\frac{1}{2}\int_{-1}^{1} I_\nu \mu^2 d\mu + \frac{1}{2}\int_{-1}^{1} \frac{\mu-\mu^3}{r}\frac{\partial I_\nu}{\partial \mu} d\mu = \frac{1}{2}\kappa S_\nu \int_{-1}^{1} \mu d\mu - \frac{1}{2}\kappa \int_{-1}^{1} I_\nu \mu d\mu$$

$$\frac{\partial}{\partial r} K_\nu + \frac{1}{2}\left[\frac{\mu-\mu^3}{r} I_\nu\right]_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} \frac{1-3\mu^2}{r} I_\nu d\mu = 0 - \kappa H_\nu$$

$$\frac{\partial}{\partial r} K_\nu + 0 - \frac{1}{r}(J_\nu - 3K_\nu) = -\kappa H_\nu$$

$$\boxed{\frac{\partial}{\partial r} K_\nu + \frac{3K_\nu}{r} - \frac{J_\nu}{r} = -\kappa H_\nu} \quad \text{(II)}$$

Solution of moment equations

Problem: n-th momentum equation contains (n+1)-st moment
$\rightarrow$ always one more unknowns than differential equations
$\rightarrow$ to close the system, another equation has to be found

Closure by introduction of variable Eddington factors

$$K_\nu = f_\nu \cdot J_\nu$$

f_ν Eddington factor, is found by iteration

$$\text{starting estimate for } f_\nu \rightarrow (I)+(II), \text{ solve} \rightarrow K_\nu$$
$$\rightarrow f_\nu^{\text{new}} = K_\nu / J_\nu$$

Solution of moment equations

$$\left.\begin{aligned}(I)\quad & \frac{dH_\nu}{d\tau} = J_\nu - S_\nu \\ (II)\quad & \frac{d(f_\nu J_\nu)}{d\tau} = H_\nu\end{aligned}\right\}\text{ 2 differential eqs. for } J_\nu, H_\nu$$

Start: approximation for f_ν , assumption: anisotropy small, i.e. substitute I_ν by J_ν (Eddington approximation)

$$K_\nu(\tau) = \frac{1}{2}\int_{-1}^{1} I_\nu \mu^2 d\mu \approx J_\nu \frac{1}{2}\int_{-1}^{1} \mu^2 d\mu = J_\nu \frac{1}{2}\left[\frac{1}{3}\mu^3\right]_{-1}^{1} = J_\nu \frac{1}{3}$$

$$\rightarrow K_\nu = \frac{1}{3} J_\nu$$

$$\rightarrow f_\nu = \frac{1}{3}$$

Eddington approximation

Is exact, if I_ν linear in μ

(one can show by Taylor expansion of S in terms of B that this linear relation is very good at large optical depths)

$$I_\nu(\mu) = I_{0\nu} + \mu I_{1\nu}$$

$$J_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu) d\mu = I_{0\nu}$$

$$H_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu)\mu d\mu = \frac{1}{2}\left[I_{1\nu}\frac{\mu^3}{3}\right]_{-1}^{1} = \frac{1}{3} I_{1\nu}$$

$$K_\nu = \frac{1}{2}\int_{-1}^{1} I_\nu(\mu)\mu^2 d\mu = \frac{1}{2}\left[I_{0\nu}\frac{\mu^3}{3}\right]_{-1}^{1} = \frac{1}{3} I_{0\nu}$$

$$\boxed{\Rightarrow K_\nu = \frac{1}{3} J_\nu}$$

Summary: Radiation Transfer

Transfer equation $\frac{dI_\nu}{ds} = \eta_\nu - \kappa(\nu) I_\nu$

Emission and absorption coefficients $\eta_\nu \; , \; \kappa(\nu)$

Definitions: source function $S_\nu = \eta_\nu / \kappa(\nu)$
optical depth $d\tau = \kappa \cdot ds$

Formal solution of transfer equation

$$I_\nu^+(\tau) = \int_\tau^{\tau_{\max}} S_\nu(\tau') \exp\left(-\frac{\tau' - \tau}{\mu}\right) \frac{d\tau'}{\mu} + I_\nu^+(\tau_{\max}) \exp\left(-\frac{\tau_{\max} - \tau}{\mu}\right)$$

Eddington-Barbier relation $I_\nu^+(0) \approx S_\nu(\tau = \mu)$

LTE **L**ocal **T**hermodynamic **E**quilibrium

$S_\nu(\nu, \vec{r}) = B_\nu(\nu, T(\vec{r}))$ $\quad T(\vec{r})$ local temperature

Including scattering: $S_\nu = \rho J_\nu + (1-\rho) B_\nu \quad \text{with} \quad \rho = \sigma / (\sigma + \chi)$

Schwarzschild-Milne equations

Moment equations of formal solution

$$J_\nu(\tau) = \Lambda(S_\nu) \qquad H_\nu(\tau) = \frac{1}{4}\Phi(S_\nu) \qquad \Lambda,\ \Phi \quad \text{integral operators}$$

Moments of transfer equation (plane-parallel)

$$\frac{dH_\nu}{d\tau} = J_\nu - S_\nu \qquad \frac{d(K_\nu)}{d\tau} = H_\nu$$

Differential equation system (for J,H,K), closed by variable Eddington factor

$$f_\nu := K_\nu / J_\nu$$

Summary: How to calculate I and the moments J,H,K (with given source function S)?

Solve transfer equation $\frac{dI}{d\tau} = \frac{1}{\mu}(I - S)$ (no irradiation from outside, semi-infinite atmosphere, drop frequency index)

Formal solution: $I^+(\tau) = \int_\tau^\infty S(\tau')\exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu} \quad (\mu > 0), \quad I^- \text{ analogous}$

How to calculate the higher moments? Two possibilities:

1. Insert formal solution into definitions of J,H,K: $\frac{1}{2}\int_{-1}^{1} I\mu^n d\mu$

$$\rightarrow J(\tau) = \Lambda(S) \quad H(\tau) = \frac{1}{4}\Phi(S) \quad K(\tau) = \frac{1}{4}\mathrm{X}(S)$$ Schwarzschild-Milne equations

2. Angular integration of transfer equation, i.e. 0-th & 1st moment $\frac{1}{2}\int_{-1}^{1} ...\mu^n d\mu$

$$\rightarrow \frac{dH}{d\tau} = J - S \qquad \frac{d(K)}{d\tau} = H$$ 2 moment equations for 3 quantities J,H,K

Eliminate K by Eddington factor f: $K = f \cdot J$

$$\rightarrow \boxed{\frac{dH}{d\tau} = J - S \qquad \frac{d(f\cdot J)}{d\tau} = H \quad \text{solve: J,H,K} \leftrightarrows \text{new f (=K/J)}}$$ iteration

Emission and Absorption

Chemical composition

Stellar atmosphere = mixture, composed of many chemical elements, present as atoms, ions, or molecules

Abundances, e.g., given as mass fractions β_k

- Solar abundances

$$\left.\begin{aligned} \beta_H &= 0.71 \\ \beta_{He} &= 0.28 \\ \beta_C &= 0.004 \\ \beta_N &= 0.001 \\ \beta_O &= 0.009 \\ &\vdots \\ \beta_{Fe} &= 0.001 \\ &\vdots \end{aligned}\right\}$$

Universal abundance for Population I stars

Chemical composition

- Population II stars

$$\beta_H = \beta_H^{\odot}$$
$$\beta_{He} = \beta_{He}^{\odot}$$
$$\beta_Z = 0.1 \cdots 0.00001\ \beta_Z^{\odot}$$

- Chemically peculiar stars, e.g., helium stars

$$\beta_H \le 0.002 << \beta_H^{\odot}$$
$$\beta_{He} = 0.964 >> \beta_{He}^{\odot}$$
$$\beta_C = 0.029 >> \beta_C^{\odot}$$
$$\beta_N = 0.003 \approx \beta_N^{\odot}$$
$$\beta_O = 0.002 < \beta_O^{\odot}$$

- Chemically peculiar stars, e.g., PG1159 stars

$$\beta_H \le 0.05 << \beta_H^{\odot}$$
$$\beta_{He} = 0.25 >> \beta_{He}^{\odot}$$
$$\beta_C = 0.55 >> \beta_C^{\odot}$$
$$\beta_N < 0.02$$
$$\beta_O = 0.15 >> \beta_O^{\odot}$$

Other definitions

- Particle number density N_k = number of atoms/ions of element k per unit volume
  relation to mass density:

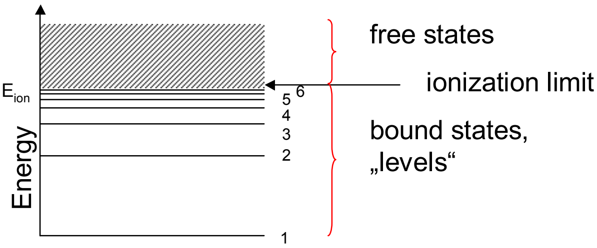
$$\beta_k \rho = A_k m_H N_k$$

  with A_k = mean mass of element k in atomic mass units (AMU)
  m_H = mass of hydrogen atom
- Particle number fraction $\dfrac{N_k}{\sum N_{k'}}$
- logarithmic $\varepsilon_k = \log(N_k / N_H) + 12.00$
- Number of atoms per 10^6 Si atoms (meteorites)

The model atom

The population numbers (=occupation numbers)

n_i = number density of atoms/ions of an element, which are in the level i



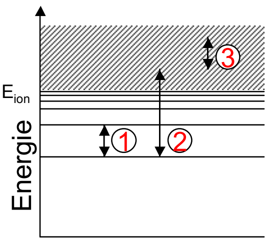
E_i = energy levels, quantized

E_1 = E(ground state) = 0

E_{ion} = ionization energy

Photon absorption cross-sections

Transitions in atoms/ions



1. bound-bound transitions = lines
2. bound-free transitions = ionization and recombination processes
3. free-free transitions = Bremsstrahlung

We look for a relation between macroscopic quantities $\kappa(\nu), \eta_\nu(\nu)$ and microscopic (quantum mechanical) quantities, which describe the state transitions within an atom

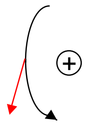
Photon absorption cross-sections

Line transitions: $\Delta E_{\mathrm{bb}} = \pm\left(E_{\mathrm{up}} - E_{\mathrm{low}}\right)$

Bound-free transitions: thermal average of electron velocities v (Maxwell distribution, i.e., electrons in thermodynamic equilibrium)

$$\text{unbound state} \;=\; \text{ion} \;+\; \text{free electron} \;\left(1/2\;\; m_e v^2\right)$$

$$\left|\Delta E_{\mathrm{bf}}\right| > E_{\mathrm{th}} = E_{\mathrm{ion}} - E_{\mathrm{low}}$$



Free-free transition: free electron in Coulomb field of an ion, Bremsstrahlung, classically: jump into other hyperbolic orbit, ΔE_{ff} arbitrary

For all processes holds: ΔE can only be supplied or removed by:

- Inelastic collisions with other particles (mostly electrons), **collisional processes**
- By absorption/emission of a photon, **radiative processes**
- In addition: **scattering processes** = (in)elastic collisions of photons with electrons or atoms
  - scattering off free electrons: Thomson or Compton scattering
  - scattering off bound electrons: Rayleigh scattering

The line absorption cross-section

Classical description (H.A. Lorentz)

Harmonic oscillator in electromagnetic field

- Damped oscillations (1-dim), eigen-frequency ω_0
  Damping constant γ
- Periodic excitation with frequency ω by E-field

Equation of motion:

$$m\ddot{x} + \gamma m\dot{x} + m\omega_0^2 x = eE_0 e^{i\omega t}$$

inertia + damping + restoring force = excitation

Usual Ansatz for solution: $x(t) = x_0 e^{i\omega t}$

$$\left(-\omega^2 + i\omega\gamma + \omega_0^2\right)x = \frac{eE_0}{m}\,e^{i\omega t}$$

The line absorption cross-section

$$\left(-\omega^2 + i\omega\gamma + \omega_0^2\right) x(t) = \frac{eE_0}{m} e^{i\omega t}$$

$$x(t) = \frac{eE_0}{m} e^{i\omega t} \cdot \frac{1}{\left(\omega_0^2 - \omega^2 + i\omega\gamma\right)}$$

expand $$x(t) = \frac{eE_0}{m} e^{i\omega t} \cdot \frac{\left(\omega_0^2 - \omega^2 - i\omega\gamma\right)}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

real part $$\mathrm{Re}(x(t)) = \frac{eE_0}{m}\left[\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\cos\omega t + \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\sin\omega t\right]$$

Electrodynamics: radiated power

$$p(t) = \frac{2}{3}\frac{e^2}{c^3}(\ddot{x})^2$$

$$\ddot{x}(t) = \frac{eE_0}{m}\left[\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}(-\omega^2)\cos\omega t + \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}(-\omega^2)\sin\omega t\right]$$

$$(\ddot{x}(t))^2 = \left(\frac{eE_0}{m}\right)^2\left[\frac{\left(\omega_0^2 - \omega^2\right)^2\omega^4}{N^2}\cos^2\omega t + \frac{2\gamma\left(\omega_0^2 - \omega^2\right)\omega^5}{N^2}\cos\omega t\sin\omega t + \frac{\gamma^2\omega^6}{N^2}\sin^2\omega t\right]$$

The line absorption coss-section

average over one period

$$\overline{\cos^2\omega t} = \overline{\sin^2\omega t} = 1/2, \quad \overline{\cos\omega t\sin\omega t} = 0$$

$$\overline{(\ddot{x})^2} = \frac{1}{2}\left(\frac{eE_0\omega^2}{m}\right)^2\left[\frac{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}{\left(\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right)^2}\right]$$

$$\overline{(\ddot{x})^2} = \frac{1}{2}\left(\frac{eE_0}{m}\right)^2\left[\frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}\right]$$

power, averaged over one period

$$\overline{p} = \frac{2}{3}\frac{e^2}{c^3}\overline{(\ddot{x})^2} = \left(\frac{e^4E_0^2}{3m^2c^3}\right)\left[\frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2}\right]$$

$$\overline{p} = \frac{e^4E_0^2}{3m^2c^3}\varphi(\nu)/C$$ C=normalization constant $(\nu = \omega/2\pi)$

$$\varphi(\nu) = \frac{\nu^4 C}{\left(\nu_0^2 - \nu^2\right)^2 + (\gamma/2\pi)^2\nu^2}$$ profile function

The line absorption cross-section

since $\Delta\nu = \nu\text{-}\nu_0 << \nu, \nu_0$: $\quad \nu \approx \nu_0$

$$(\nu_0^2 - \nu^2)^2 = ((\nu_0 + \nu)(\nu_0 - \nu))^2 \approx 4\nu_0^2(\nu_0 - \nu)^2$$

$$\varphi(\nu) = \frac{\nu_0^2 C}{4(\nu_0 - \nu)^2 + (\gamma/2\pi)^2} = \frac{C}{4}\frac{\nu_0^2}{(\nu_0 - \nu)^2 + (\gamma/4\pi)^2}$$

now: calculating the normalization constant

$$\int_{\nu_0-\infty}^{\nu_0+\infty} \varphi(\nu)d\nu = 1$$

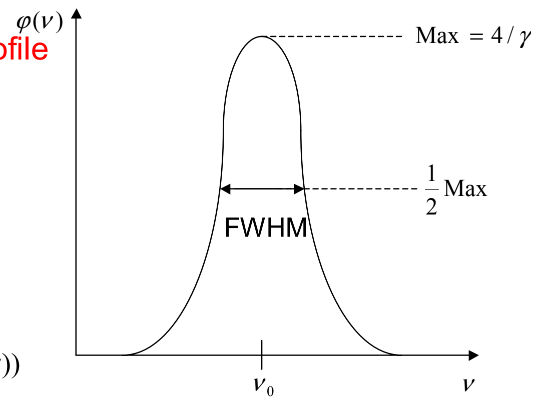
substitution: $\quad x := \frac{4\pi}{\gamma}(\nu_0 - \nu)$

$$\int_{\nu_0-\infty}^{\nu_0+\infty} \varphi(\nu)d\nu = \frac{C}{4}\nu_0^2\frac{4\pi}{\gamma}\underbrace{\int_{-\infty}^{+\infty}\frac{dx}{1+x^2}}_{=\pi} = C\frac{\nu_0^2\pi^2}{\gamma} \Rightarrow C = \frac{\gamma}{\nu_0^2\pi^2}$$

The line absorption cross-section

Profile function, Lorentz profile

$$\boxed{\varphi(\nu) = \frac{\gamma/4\pi^2}{(\nu_0 - \nu)^2 + (\gamma/4\pi)^2}}$$



properties:

- Symmetry:

$$\varphi(+(\nu_0 - \nu)) = \varphi(-(\nu_0 - \nu))$$

- Asymptotically:

$$\varphi(\nu) \rightarrow 1/(\nu_0 - \nu)^2 = 1/\Delta\nu^2$$

- FWHM:

$$\frac{2}{\gamma} = \frac{\gamma/4\pi^2}{(\Delta\nu_{\text{FWHM}}/2)^2 + (\gamma/4\pi)^2} \Rightarrow \Delta\nu_{\text{FWHM}} = \frac{2\gamma}{4\pi} = \frac{\gamma}{2\pi}$$

The damping constant

- Radiation damping, classically (other damping mechanisms later)
- Damping force ("friction") $F = \gamma m \dot{x}(t)$
  power=force ·velocity $p(t) = \gamma m(\dot{x}(t))^2$
  electrodynamics $p(t) = \frac{2}{3}\frac{e^2}{c^3}(\ddot{x}(t))^2$
- Hence, Ansatz for frictional force is not correct
- Help: define γ such, that the power is correct, when time-averaged over one period: $\gamma m \omega^2 = \frac{2}{3}\frac{e^2}{c^3}\omega^4$ (where we used $x(t) = x_0 e^{i\omega t}$)

$\omega \approx \omega_0 \Rightarrow \boxed{\gamma = \frac{2}{3}\frac{e^2\omega_0^2}{mc^3}}$ classical radiation damping constant

Half-width

Insert into expression for FWHM:

$$\Delta\nu_{\mathrm{FWHM}} = \frac{\gamma}{2\pi} = \frac{4\pi e^2 \nu_0^2}{3mc^3}$$

$$\frac{\Delta\nu_{\mathrm{FWHM}}}{\nu} = \frac{\Delta\lambda_{\mathrm{FWHM}}}{\lambda} \Rightarrow \Delta\lambda_{\mathrm{FWHM}} = \frac{c}{\nu^2}\Delta\nu_{\mathrm{FWHM}} = \frac{4\pi e^2}{3mc^2} = \boxed{1.18\cdot 10^{-4}\,\text{Å}}$$

The absorption cross-section

Definition absorption coefficient κ $\quad dI_\nu = -\kappa(\nu) I_\nu ds$

with n_{low} = number density of absorbers: $\kappa(\nu) = \sigma(\nu) n_{\mathrm{low}}$

$\sigma(\nu)$ absorption cross-section (definition), dimension: area

Separating off frequency dependence: $\sigma(\nu) = \sigma_0 \varphi(\nu)$

Dimension σ_0: area · frequency

Now: calculate absorption cross-section of classical harmonic oscillator for plane electromagnetic wave:

$$E_x = E_0 e^{i\omega t}$$

$$I_\nu(\nu', \mu) = \frac{c}{8\pi} E_0^2 \delta(\nu - \nu') \delta(\mu - 1)$$

Stellar Atmospheres: Emission and Absorption

Power, averaged over one period, extracted from the radiation field:

$$\bar{p} = \frac{e^4 E_0^2}{3m^2c^3} \frac{\pi^2 \nu_0^2}{\gamma} \varphi(\nu) \quad \text{with} \quad \gamma = \gamma_{\text{class.}} = \frac{2}{3} \frac{e^2 \omega_0^2}{mc^3}$$

$$\bar{p} = \frac{e^4 E_0^2}{3m^2c^3} \frac{\pi^2 \nu_0^2 3mc^3}{2e^2 4\pi^2 \nu_0^2} \varphi(\nu) = \frac{e^2 E_0^2}{8m} \varphi(\nu)$$

On the other hand: $\bar{p} = \sigma(\nu) \int_{\nu'} \int_{\mu} I(\nu', \mu) d\nu' d\mu = \sigma(\nu) \frac{c}{8\pi} E_0^2$

Equating: $\sigma(\nu) \frac{c}{8\pi} E_0^2 = \frac{e^2 E_0^2}{8m} \varphi(\nu)$

$$\sigma(\nu) = \frac{\pi e^2}{mc} \varphi(\nu) \Rightarrow \sigma_0 = 0.026537 \text{ cm}^2 \text{ Hz}$$

Classically: independent of particular transition

Quantum mechanically: correction factor, oscillator strength

$$\boxed{\sigma_{\mathrm{lu}} = \frac{\pi e^2}{mc} f_{\mathrm{lu}} \qquad \kappa(\nu) = n_{\mathrm{low}} \frac{\pi e^2}{mc} f_{\mathrm{lu}} \varphi(\nu)}$$

index "lu" stands for transition lower→upper level

Oscillator strengths

Oscillator strengths f_{lu} are obtained by:

- Laboratory measurements
- Solar spectrum
- Quantum mechanical computations (Opacity Project etc.)

λ/Å	Line	f_{lu}	g_{low}	g_{up}
1215.7	Ly α	0.41	2	8
1025.7	Ly β	0.07	2	18
972.5	Ly γ	0.03	2	32
6562.8	H α	0.64	8	18
4861.3	H β	0.12	8	32
4340.5	H γ	0.04	8	50

- Allowed lines: $f_{lu}\approx 1$,
- Forbidden: <<1 e.g. He I $1s^2\,{}^1S \rightarrow 1s2s\,{}^3S$ $f_{lu}=2\ 10^{-14}$

Opacity status report

Connecting the (macroscopic) opacity with (microscopic) atomic physics

Classical crossection

$$\kappa(\nu) = n_{\mathrm{low}} \underbrace{\frac{\pi e^2}{mc} f_{\mathrm{low,up}}}_{\sigma_{low,up}} \varphi(\nu)$$

Profile function

QM correction factor

Population number of lower level

View atoms as harmonic oscillator

- Eigenfrequency: transition energy
- Profile function: reaction of an oscillator to extrenal driving (EM wave)
- Classical crossection: radiated power = damping

Extension to emission coefficient

Alternative formulation by defining Einstein coefficients:

$$\kappa(\nu) = n_{\text{low}} \frac{h\nu_0}{4\pi} B_{\text{lu}} \varphi(\nu)$$

$$\text{i.e.} \qquad \frac{h\nu_0}{4\pi} B_{\text{lu}} = \frac{\pi e^2}{mc} f_{\text{lu}}$$

Similar definition for emission processes:

$$\eta_\nu^{\text{induced}} = n_{\text{up}} \frac{h\nu_0}{4\pi} B_{\text{ul}} I_\nu \psi(\nu)$$

$$\eta_\nu^{\text{spontaneous}} = n_{\text{up}} \frac{h\nu_0}{4\pi} A_{\text{ul}} \psi(\nu)$$

$\psi(\nu)$ profile function, complete redistribution: $\varphi(\nu) = \psi(\nu)$

Relations between Einstein coefficients

Derivation in TE; since they are atomic constants, these relations are valid independent of thermodynamic state

In TE, each process is in equilibrium with its inverse, i.e., within one line there is no netto destruction or creation of photons (detailed balance)

$$\text{emitted intensity} = \text{absorbed intensity}$$

$$B_{\text{ul}} \frac{h\nu_0}{4\pi} I_\nu n_{\text{up}} + A_{\text{ul}} \frac{h\nu_0}{4\pi} n_{\text{up}} = B_{\text{lu}} \frac{h\nu_0}{4\pi} I_\nu n_{\text{low}} \qquad \text{TE: } I_\nu = B_\nu(T)$$

$$\left(B_{\text{ul}} B_\nu(T) + A_{\text{ul}}\right) n_{\text{up}} = B_{\text{lu}} B_\nu(T) n_{\text{low}}$$

$$B_\nu(T)\left(n_{\text{low}} B_{\text{lu}} - n_{\text{up}} B_{\text{ul}}\right) = n_{\text{up}} A_{\text{ul}}$$

$$B_\nu(T) = \frac{A_{\text{ul}}}{B_{\text{ul}}} \left(\frac{n_{\text{low}} B_{\text{lu}}}{n_{\text{up}} B_{\text{ul}}} - 1 \right)^{-1}$$

Relations between Einstein coefficients

$$B_\nu(T) = \frac{A_{\mathrm{ul}}}{B_{\mathrm{ul}}}\left(\frac{n_{\mathrm{low}} B_{\mathrm{lu}}}{n_{\mathrm{up}} B_{\mathrm{ul}}} - 1\right)^{-1} \quad \text{with Boltzmann equation:} \quad \frac{n_{\mathrm{up}}}{n_{\mathrm{low}}} = \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} e^{-h\nu_0/kT}$$

$$B_\nu(T) = \frac{A_{\mathrm{ul}}}{B_{\mathrm{ul}}}\left(\frac{g_{\mathrm{low}} B_{\mathrm{lu}}}{g_{\mathrm{up}} B_{\mathrm{ul}}} e^{h\nu_0/kT} - 1\right)^{-1} \quad \text{comparison with Planck blackbody radiation:}$$

$$B_\nu(T) = \frac{2h\nu_0^3}{c^2}\left(e^{h\nu_0/kT} - 1\right)^{-1}$$

$$\boxed{\begin{aligned} &\Rightarrow \frac{A_{\mathrm{ul}}}{B_{\mathrm{ul}}} = \frac{2h\nu_0^3}{c^2} \\ &\Rightarrow \frac{g_{\mathrm{low}} B_{\mathrm{lu}}}{g_{\mathrm{up}} B_{\mathrm{ul}}} = 1 \Rightarrow g_{\mathrm{low}} B_{\mathrm{lu}} = g_{\mathrm{up}} B_{\mathrm{ul}} \end{aligned}}$$

☛

Relation to oscillator strength

$$B_{\mathrm{lu}} = \frac{4e^2\pi^2}{mch\nu_0} f_{\mathrm{lu}}$$

$$B_{\mathrm{ul}} = \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} B_{\mathrm{lu}} = \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} \frac{4e^2\pi^2}{mch\nu_0} f_{\mathrm{lu}}$$

$$A_{\mathrm{ul}} = \frac{2h\nu_0^3}{c^2} B_{\mathrm{ul}} = \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} \frac{8e^2\nu_0^2\pi^2}{mc^3} f_{\mathrm{lu}} = 3\gamma_{\mathrm{ul}} \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} f_{\mathrm{lu}}$$ dimension A_{ul} $1/\mathrm{time}$ ☚

Interpretation of $1/A_{\mathrm{ul}}$ as lifetime of the excited state

order of magnitude: $A_{\mathrm{ul}} \approx \gamma_{\mathrm{ul}}$
at 5000 Å: $10^8\,\mathrm{s}^{-1}$
lifetime: $10^{-8}\,\mathrm{s}$

Comparison induced/spontaneous emission

When is spontaneous or induced emission stronger?

with $I_\nu = B_\nu$

$$\frac{\eta_\nu^{\text{spontaneous}}}{\eta_\nu^{\text{induced}}} = \frac{A_{\text{ul}} h\nu_* n_{\text{up}}\,\psi(\nu)/4\pi}{B_\nu(T^*) B_{\text{ul}} h\nu_* n_{\text{up}}\,\psi(\nu)/4\pi} = \frac{A_{\text{ul}}}{B_{\text{ul}}}\frac{1}{B_\nu(T^*)} = \frac{2h\nu_*^3}{c^2}\frac{c^2}{2h\nu_*^3}\left(e^{h\nu_*/kT^*}-1\right)$$

$$:=1 \Rightarrow e^{h\nu_*/kT^*} = 2 \Rightarrow h\nu_*/kT^* = \ln 2$$

$$\text{e.g. } T^* = 10000\,\text{K}: \lambda_* = 20000\,\mathring{\text{A}}$$

$$T^* = 50000\,\text{K}: \lambda_* = 4160\,\mathring{\text{A}}$$

At wavelengths shorter than λ_* spontaneous emission is dominant

Induced emission as negative absorption

Radiation transfer equation:

$$\frac{dI_\nu}{ds} = \eta_\nu - \kappa I_\nu \quad \text{with } \eta_\nu = \eta_\nu^{\text{spontaneous}} + \eta_\nu^{\text{induced}}$$

$$\frac{dI_\nu}{ds} = \eta_\nu^{\text{spontaneous}} + \eta_\nu^{\text{induced}} - \kappa I_\nu$$

transition low→up $\quad \kappa_{\text{lu}} = B_{\text{lu}}\frac{h\nu_0}{4\pi} n_{\text{low}}\varphi(\nu), \quad \eta_{\text{lu}}^{\text{induced}} = B_{\text{ul}}\frac{h\nu_0}{4\pi} n_{\text{up}} I_\nu \varphi(\nu)$

Useful definition: κ corrected for induced emission:

$$\frac{dI_\nu}{ds} = \eta_\nu^{\text{spontaneous}} + \left(B_{\text{ul}} n_{\text{up}} - B_{\text{lu}} n_{\text{low}}\right)\frac{h\nu_0}{4\pi}\varphi(\nu) I_\nu$$

So we get (formulated with oscillator strength instead of Einstein coefficients):

$$\boxed{\begin{aligned} \kappa_{\text{lu}} &= \frac{\pi e^2}{mc} f_{\text{lu}}\left(n_{\text{low}} - \frac{g_{\text{low}}}{g_{\text{up}}} n_{\text{up}}\right)\varphi(\nu) \\ \eta_{\text{lu}}^{\text{spontaneous}} &= \frac{2h\nu_0^3}{c^2}\frac{\pi e^2}{mc} f_{\text{lu}}\frac{g_{\text{low}}}{g_{\text{up}}} n_{\text{up}}\varphi(\nu) \end{aligned}}$$

The line source function

General source function: $S_\nu = \eta_\nu / \kappa$

Special case: emission and absorption by one line transition:

$$S_\nu^{\mathrm{lu}} = \frac{\eta_\nu^{\mathrm{lu}}}{\kappa^{\mathrm{lu}}} = \frac{A_{\mathrm{ul}} n_{\mathrm{up}} \frac{h\nu_0}{4\pi}\varphi(\nu)}{\left(B_{\mathrm{lu}} n_{\mathrm{low}} - B_{\mathrm{ul}} n_{\mathrm{up}}\right)\frac{h\nu_0}{4\pi}\varphi(\nu)} = \frac{2h\nu_0^3}{c^2}\frac{n_{\mathrm{up}}}{\frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} n_{\mathrm{low}} - n_{\mathrm{up}}}$$

$$\boxed{S_\nu^{\mathrm{lu}} = \frac{2h\nu_0^3}{c^2}\left[\frac{g_{\mathrm{up}}}{g_{\mathrm{low}}}\frac{n_{\mathrm{low}}}{n_{\mathrm{up}}} - 1\right]^{-1}}$$

- Not dependent on frequency
- Only a function of population numbers
- In LTE:

$$S_\nu^{\mathrm{lu}} = \frac{2h\nu_0^3}{c^2}\left[e^{h\nu_0/kT} - 1\right]^{-1} = B_\nu(\nu_0, T)$$

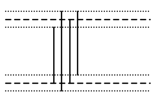
Line broadening: Radiation damping

Every energy level has a finite lifetime τ against radiative decay (except ground level) $\tau = 1 \Big/ \sum_{\mathrm{l<u}} A_{\mathrm{ul}}$

Heisenberg uncertainty principle: $\Delta E \cdot \tau = \hbar$

Energy level not infinitely sharp

q.m. $\Rightarrow$ profile function = Lorentz profile



$$\gamma = \frac{1}{\tau_{\mathrm{u}}} + \frac{1}{\tau_l} = \sum_{\mathrm{k<u}} A_{\mathrm{uk}} + \sum_{\mathrm{j}<l} A_{l\mathrm{j}}$$

Simple case: resonance lines (transitions to ground state)

example Lyα (transition 2$\rightarrow$1): $\gamma = \mathrm{A}_{21} = 3\gamma_{\mathrm{cl}}\, \mathrm{g}_1/\mathrm{g}_2\, \mathrm{f}_{12} = 3\gamma_{\mathrm{cl}}\, 2/8 \cdot 0.41 = 0.31\gamma_{\mathrm{cl}}$

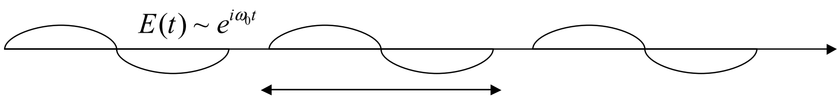
example Hα (3$\rightarrow$2):

$$\gamma = 3\gamma_{\mathrm{cl}}\left(\frac{\mathrm{g}_1}{\mathrm{g}_2}\mathrm{f}_{12} + \frac{\mathrm{g}_2}{\mathrm{g}_3}\mathrm{f}_{23} + \frac{\mathrm{g}_1}{\mathrm{g}_3}\mathrm{f}_{13}\right) = 3\gamma_{\mathrm{cl}}\left(\frac{2}{8}0.41 + \frac{8}{18}0.64 + \frac{2}{18}0.07\right) = 1.18\gamma_{\mathrm{cl}}$$

Line broadening: Pressure broadening

Reason: collision of radiating atom with other particles

⇒Phase changes, disturbed oscillation

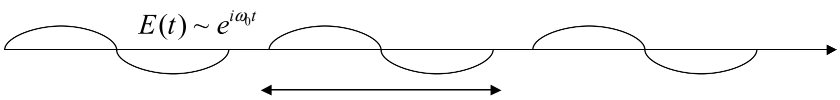


t_0 = time between two collisions

Line broadening: Pressure broadening

Reason: collision of radiating atom with other particles

⇒Phase changes, disturbed oscillation



t_0 = time between two collisions

Intensity spectrum (=power spectrum) of the cut wave train:

$$I_1 \sim |\text{Fourier transform}|^2$$

$$I_1(\omega) \sim \left| \int_{-t_0/2}^{t_0/2} e^{i\omega_0 t} e^{i\omega t} dt \right|^2 = \left[\frac{\sin\left(\frac{\omega-\omega_0}{2} t\right)}{\frac{\omega-\omega_0}{2}} \right]^2$$

Line broadening: Pressure broadening

Probability distribution for t_0

$$W(t_0)dt_0 = e^{-t_0/\tau}\left(dt_0/\tau\right) \qquad \tau = \text{average time between two collisions}$$

Averaging over all t_0 gives

$$I_\nu(\omega) = \text{const}\cdot\int_0^\infty\left[\sin\left(\frac{\omega-\omega_0}{2}t\right)\Big/\frac{\omega-\omega_0}{2}\right]^2 e^{-t_0/\tau}dt_0/\tau$$

Performing integration and normalization gives profile function of intensity spectrum: $\varphi(\omega) = \dfrac{1/\pi\tau}{(\omega-\omega_0)^2+(1/\tau)^2}$

i.e. profile function for collisional broadening is a Lorentz profile with

$$\gamma = 2/\tau,\ \ \tau \sim N^{-1} \qquad N = \text{particle density of colliders}$$
$$\gamma = N\cdot\gamma' \qquad \gamma' \text{ approximately constant}$$

(to calculate γ': calculation of τ necessary; for that: assumption about phase shift needed, e.g., given by semi-classical theory)

Line broadening: Pressure broadening

- Semi-classical theory (Weisskopf, Lindholm), „Impact Theory“
  Phase shifts $\Delta\omega$:
  $$\text{Ansatz: } \Delta\omega = C_p/r^p,\ \ p = 2,3,4,6\ ,\ r(t) = \text{distance to colliding particle}$$
  find constants C_p by laboratory measurements, or calculate

p=	name	dominant at
2	linear Stark effect	hydrogen-like ions
3	resonance broadening	neutral atoms with each other, H+H
4	quadratic Stark effect	ions
6	van der Waals broadening	metals + H

- Good results for p=2 (H, He II): „Unified Theory“
  – H Vidal, Cooper, Smith 1973
  – He II Schöning, Butler 1989
- For p=4 (He I)
  – Barnard, Cooper, Shamey; Barnard, Cooper, Smith; Beauchamp et al.

Film logg

Thermal broadening

Thermal motion of atoms (Doppler effect)

Velocities distributed according to Maxwell, i.e.

$$w_x(\mathrm{v}_x) \sim e^{-1/2 m_A \mathrm{v}_x^2 / kT}$$

for one spatial direction x (line-of-sight)

Thermal (most probable) velocity v_{th}:

$$\mathrm{v}_{\mathrm{th}} = \sqrt{2kT/m_A} = 12.85\left(T/10^4\,\mathrm{A}\right)^{1/2}\ \mathrm{km/s}$$

example: $\mathrm{T} = 6000\mathrm{K},\ \mathrm{A} = 56$ (iron): $\mathrm{v}_{\mathrm{th}} = 1.33$ km/s

i.e. $w_x(\mathrm{v}_x) = C \cdot e^{-\mathrm{v}_x^2/\mathrm{v}_{\mathrm{th}}^2}$, with $\int_0^\infty w_x(\mathrm{v}_x)\,d\mathrm{v}_x = 1$ we obtain:

$$\mathrm{C} \cdot \int_0^\infty e^{-\mathrm{v}_x^2/\mathrm{v}_{\mathrm{th}}^2} d\mathrm{v}_x = \mathrm{C} \cdot \mathrm{v}_{\mathrm{th}} \int_0^\infty e^{-x^2} d\mathrm{x} = 1 \Rightarrow C\sqrt{\pi}\mathrm{v}_{\mathrm{th}} = 1 \Rightarrow C = \frac{1}{\sqrt{\pi}\mathrm{v}_{\mathrm{th}}}$$

$$\boxed{w_x(\mathrm{v}_x) = \frac{1}{\sqrt{\pi}\mathrm{v}_{\mathrm{th}}} e^{-\mathrm{v}_x^2/\mathrm{v}_{\mathrm{th}}^2}}$$

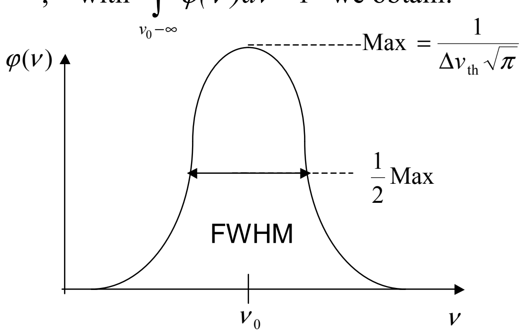
Line profile

Doppler effect: $\frac{\Delta\nu}{\nu_0} = \frac{\mathrm{v}}{c}\ ,\ \frac{\Delta\nu_{\mathrm{th}}}{\nu_0} = \frac{\mathrm{v}_{\mathrm{th}}}{c}$

profile function:

$$w_x(\mathrm{v}_x) \Rightarrow \varphi(\nu) = \frac{C_1}{\sqrt{\pi}\Delta\nu_{\mathrm{th}}} \frac{\nu_0}{c} e^{-\Delta\nu^2/\Delta\nu_{\mathrm{th}}^2},\quad \text{with} \int_{\nu_0-\infty}^{\nu_0+\infty} \varphi(\nu)\,d\nu = 1 \quad \text{we obtain:}$$

$$\boxed{\varphi(\nu) = \frac{1}{\sqrt{\pi}\Delta\nu_{\mathrm{th}}} e^{-(\nu-\nu_0)^2/\Delta\nu_{\mathrm{th}}^2}}$$



Line profile = Gauss curve

- Symmetric about ν_0
- Maximum: $1/\Delta\nu_{\mathrm{th}}\sqrt{\pi}$
- Half width: $\Delta\nu_{\mathrm{FWHM}} = 2\sqrt{\ln 2}\,\Delta\nu_{\mathrm{th}} = 1.67\,\Delta\nu_{\mathrm{th}}$
- Temperature dependency: $\Delta\nu_{\mathrm{th}} \sim \sqrt{T}$

Examples

At λ_0=5000Å:

T=6000K, A=56 (Fe): $\Delta\lambda_{th}$=0.02Å

T=50000K, A=1 (H): $\Delta\lambda_{th}$=0.5Å

Compare with radiation damping: $\Delta\lambda_{FWHM}$=1.18 10^{-4}Å

But: decline of Gauss profile in wings is much steeper than for Lorentz profile:

$$\text{Gauss } (10\Delta\lambda_{th}) \quad : \quad e^{-10^2} \approx 10^{-43}$$
$$\approx$$
$$\text{Lorentz } (1000\Delta\lambda_{rad}) : \; 1/1000^2 \approx 10^{-6}$$

In the line wings the Lorentz profile is dominant

Line broadening: Microturbulence

Reason: chaotic motion (turbulent flows) with length scales smaller than photon mean free path

Phenomenological description:

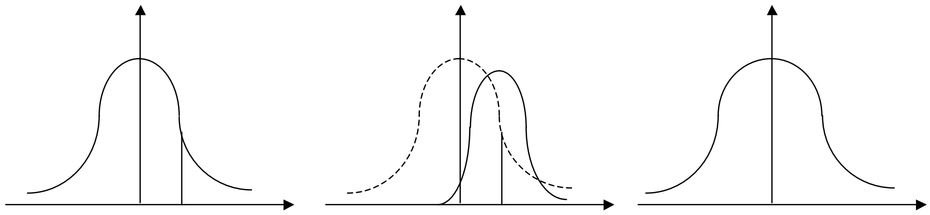
Velocity distribution: $w_x(v_x) = \dfrac{1}{\sqrt{\pi} v_{micro}} e^{-v_x^2/v_{micro}^2}$

i.e., in analogy to thermal broadening

v_{micro} is a free parameter, to be determined empirically

Solar photosphere: v_{micro} =1.3 km/s

Joint effect of different broadening mechanisms



Mathematically: convolution $(f_A * f_B)(x) = \int_{-\infty}^{\infty} f_A(y) f_B(x-y)\,dy$

commutative: $f_A * f_B = f_B * f_A$

multiplication of areas: $\int_{-\infty}^{\infty} (f_A * f_B)(x)dx = \int_{-\infty}^{\infty} f_A(x)dx \cdot \int_{-\infty}^{\infty} f_B(x)dx$

Fourier transformation: $\widetilde{f_A * f_B} = \sqrt{2\pi}\,\tilde{f}_A \cdot \tilde{f}_B$

i.e.: in Fourier space the convolution is a multiplication

Application to profile functions

Convolution of two Gauss profiles (thermal broadening + microturbulence)

$$G_A(x) = 1\big/ A\sqrt{\pi} \quad e^{-x^2/A^2} \quad G_B(x) = 1\big/ B\sqrt{\pi} \quad e^{-x^2/B^2}$$

$$\mathrm{G_C}(x) = G_A(x) * G_B(x) = 1\big/ C\sqrt{\pi} \quad e^{-x^2/C^2} \quad \text{with} \quad \mathrm{C}^2 = A^2 + B^2$$

Result: Gauss profile with quadratic summation of half-widths; proof by Fourier transformation, multiplication, and back-transformation

Convolution of two Lorentz profiles (radiation + collisional damping)

$$L_A(x) = \frac{A/\pi}{x^2 + A^2} \quad L_B(x) = \frac{B/\pi}{x^2 + B^2}$$

$$L_C(x) = L_A(x) * L_B(x) = \frac{C/\pi}{x^2 + C^2} \quad \text{with} \quad C = A + B$$

Result: Lorentz profile with sum of half-widths; proof as above

Application to profile functions

Convolving Gauss and Lorentz profile (thermal broadening + damping)

$$G(\nu)=\frac{1}{\Delta\nu_D\sqrt{\pi}}\,e^{-(\nu-\nu_0)^2/\Delta\nu_D{}^2}\qquad L(\nu)=\frac{\gamma/4\pi^2}{(\nu-\nu_0)^2+(\gamma/4\pi)^2}$$

$$V=G*L\quad\text{depends on}\quad \nu,\Delta\nu,\gamma,\Delta\nu_D:\quad V(\nu)=\int_{-\infty}^{\infty}G(\nu')L(\nu-\nu')d\nu'$$

Transformation: $\mathrm{v}:=(\nu-\nu_0)/\Delta\nu_D\quad a:=\gamma/(4\pi\Delta\nu_D)\quad y:=(\nu'-\nu_0)/\Delta\nu_D$

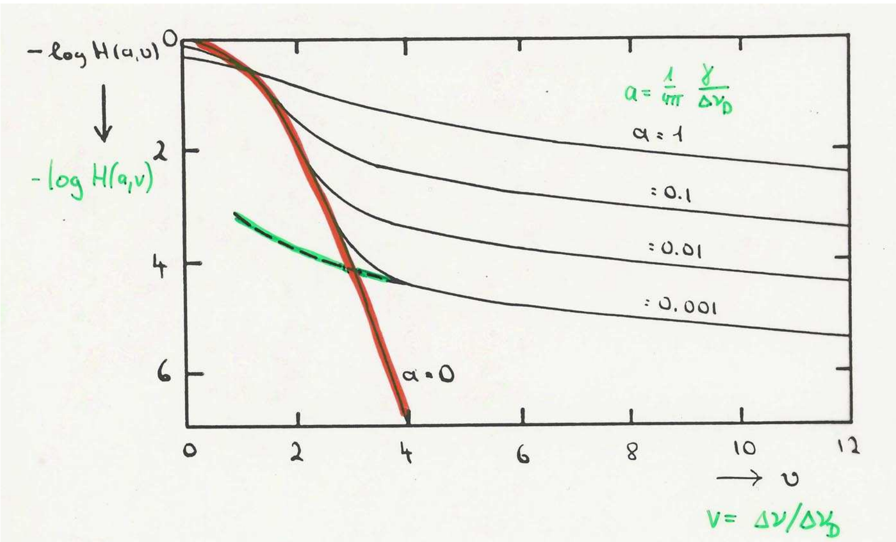
$$G(y)=\frac{1}{\Delta\nu_D\sqrt{\pi}}\,e^{-y^2}\quad L(y)=\frac{a/\Delta\nu_D\pi}{y^2+a^2}\quad V=\frac{1}{\Delta\nu_D\sqrt{\pi}}\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{-y^2}}{(\mathrm{v}-y)^2+a^2}dy$$

Def: $V=\frac{1}{\Delta\nu_D\sqrt{\pi}}H(a,\mathrm{v})$ with $\boxed{H(a,\mathrm{v})=\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{-y^2}}{(\mathrm{v}-y)^2+a^2}dy}$

Voigt function, no analytical representation possible.
(approximate formulae or numerical evaluation)

Normalization: $\int_{-\infty}^{\infty}H(a,\mathrm{v})d\mathrm{v}=\sqrt{\pi}$

Voigt profile, line wings



Treatment of very large number of lines

Example: bound-bound opacity for 50Å interval in the UV:

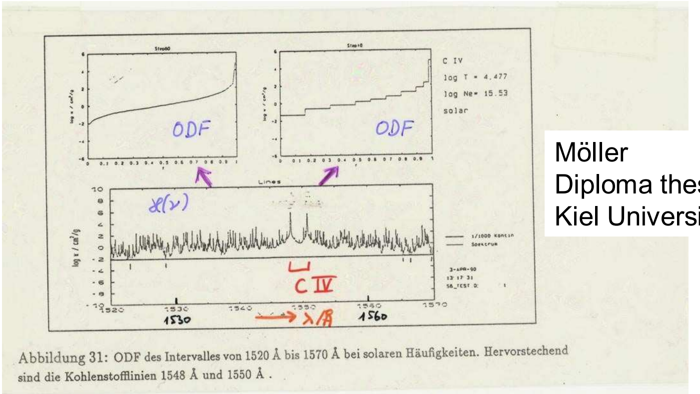


Abbildung 31: ODF des Intervalles von 1520 Å bis 1570 Å bei solaren Häufigkeiten. Hervorstechend sind die Kohlenstofflinien 1548 Å und 1550 Å .

Möller
Diploma thesis
Kiel University 1990

Direct computation would require very much frequency points

- Opacity Sampling
- Opacity Distribution Functions ODF (Kurucz 1979)

Stellar Atmospheres: Emission and Absorption

Bound-free absorption and emission

Einstein-Milne relations, Milne 1924: Generalization of Einstein relations to continuum processes: photoionization and recombination

Recombination spontaneous + induced

Transition probabilities:

P_ν : probability for photoionization in $[\nu, \nu + d\nu]$

$F(\mathrm{v})$: spontaneous recapture probability of electron in $[\mathrm{v}, \mathrm{v} + d\mathrm{v}]$

$G(\mathrm{v})$: corresponding induced probability v=electron velocity

I) number of photoionizations $n_{\text{low}} P_\nu I_\nu d\nu dt$

II) number of recombinations $n_{\text{up}} n_{\text{e}}(\mathrm{v})[F(\mathrm{v}) + G(\mathrm{v}) I_\nu]\mathrm{v} d\mathrm{v} dt$

Photon energy $h\nu = E_{\text{ion}} + 1/2\, m\mathrm{v}^2 \rightarrow d\nu = m/h\, \mathrm{v} d\mathrm{v}$

In TE, detailed balancing: I) = II)

Einstein-Milne relations

$$n_{\rm low} P_\nu I_\nu d\nu dt = n_{\rm up} n_{\rm e}({\rm v})\left[F({\rm v}) + G({\rm v}) I_\nu\right] h/m \; d{\rm v} dt \quad \text{with} \quad I_\nu = B_\nu$$

$$n_{\rm low} P_\nu B_\nu = n_{\rm up} n_{\rm e}({\rm v})\left[F({\rm v}) + G({\rm v}) B_\nu\right] h/m$$

$$B_\nu = \frac{F({\rm v})}{G({\rm v})}\left[\frac{n_{\rm low} P_\nu m}{n_{\rm up} n_{\rm e}({\rm v}) h G({\rm v})} - 1\right]^{-1} = \frac{2h\nu^3}{c^2}\left[e^{h\nu/kT} - 1\right]^{-1}$$

$$\boxed{\Rightarrow \frac{F({\rm v})}{G({\rm v})} = \frac{2h\nu^3}{c^2}}$$

$$\Rightarrow \frac{n_{\rm low} P_\nu m}{n_{\rm up} n_{\rm e}({\rm v}) h G({\rm v})} = e^{h\nu/kT}$$

- $n_{\rm low}/n_{\rm up}$ from Saha equation: $\dfrac{n_{\rm low}}{n_{\rm up}} = \dfrac{2}{n_{\rm e}}\left(\dfrac{2\pi mkT}{h^2}\right)^{3/2} \dfrac{g_{\rm up}}{g_{\rm low}} e^{-E_{\rm ion}/kT}$
- $n_{\rm e}({\rm v})$: Maxwell distribution: $n_{\rm e}({\rm v}) d{\rm v} = n_{\rm e}\left(\dfrac{m}{2\pi kT}\right)^{3/2} e^{-m{\rm v}^2/2kT} 4\pi {\rm v}^2 d{\rm v}$

Einstein-Milne relations

$$\frac{P_\nu}{G({\rm v})} = \frac{h}{m} e^{h\nu/kT} \frac{n_{\rm up}}{n_{\rm low}} n_{\rm e}({\rm v})$$

$$= \frac{h}{m} e^{h\nu/kT} \frac{2}{n_{\rm e}}\left(\frac{2\pi mkT}{h^2}\right)^{3/2} \frac{g_{\rm up}}{g_{\rm low}} e^{-E_{\rm ion}/kT} n_{\rm e}\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m{\rm v}^2/2kT} 4\pi {\rm v}^2$$

$$= \frac{h}{m} 2\left(\frac{m}{h^2}\right)^{3/2} \frac{g_{\rm up}}{g_{\rm low}} m^{3/2} 4\pi {\rm v}^2$$

$$\boxed{\frac{P_\nu}{G({\rm v})} = \frac{8\pi m^2}{h^3} \frac{g_{\rm up}}{g_{\rm low}} {\rm v}^2}$$

Einstein-Milne relations, continuum analogs to A_{ji}, B_{ji}, B_{ij}

Absorption and emission coefficients

absorption coefficient (opacity) $\kappa(\nu) = n_{\mathrm{low}} P_\nu h\nu = n_{\mathrm{low}} \sigma_\nu$ definition. of cross-section σ

emission coefficient (emissivity) $\eta_\nu(\nu) = n_{\mathrm{up}} n_{\mathrm{e}}(\mathrm{v})[F(\mathrm{v}) + G(\mathrm{v}) I_\nu] h^2 \nu / m$

And again: induced emission as negative absorption

$$\kappa(\nu) = n_{\mathrm{low}} P_\nu h\nu - n_{\mathrm{up}} n_{\mathrm{e}}(\mathrm{v}) G(\mathrm{v}) h\nu^2 / m$$

$$= n_{\mathrm{low}} P_\nu h\nu \left[1 - \frac{n_{\mathrm{up}}}{n_{\mathrm{low}}} n_{\mathrm{e}}(\mathrm{v}) \frac{G(\mathrm{v})}{P_\nu} \frac{h}{m}\right] = \ldots = \sigma_\nu \left[n_{\mathrm{low}} - n_{\mathrm{up}} \left(\frac{n_{\mathrm{up}}}{n_{\mathrm{low}}}\right)^* e^{-h\nu/kT}\right]$$

and $\eta_\nu(\nu) = \ldots = \frac{2h\nu}{\mathrm{c}^3} \sigma_\nu n_{\mathrm{up}} \left(\frac{n_{\mathrm{up}}}{n_{\mathrm{low}}}\right)^* e^{-h\nu/kT}$ (using Einstein-Milne relations)

LTE: $\kappa(\nu) = n_{\mathrm{low}} P_\nu h\nu \left[1 - e^{-h\nu/kT}\right]$

$$\eta_\nu(\nu) = n_{\mathrm{up}} n_{\mathrm{e}}(\mathrm{v}) F(\mathrm{v}) h\mathrm{v}^2 / m$$

$$\vdots$$

$$\eta_\nu(\nu) = \kappa(\nu) B_\nu$$

Continuum absorption cross-sections

H-like ions: semi-classical Kramers formula

$$\sigma(\nu) = \begin{cases} \sigma_{\mathrm{th}} (\nu_{\mathrm{th}}/\nu)^3 & \text{for } \nu > \nu_{\mathrm{th}} \\ 0 & \text{else} \end{cases}$$

ν_{th} = threshold frequency, $\sigma_{\mathrm{th}} = \frac{8h^3}{3\sqrt{3}\pi^2 m^2 c e^2} \frac{n}{Z^2} = 7.906 \cdot 10^{-18} \mathrm{cm}^2 \frac{n}{Z^2}$

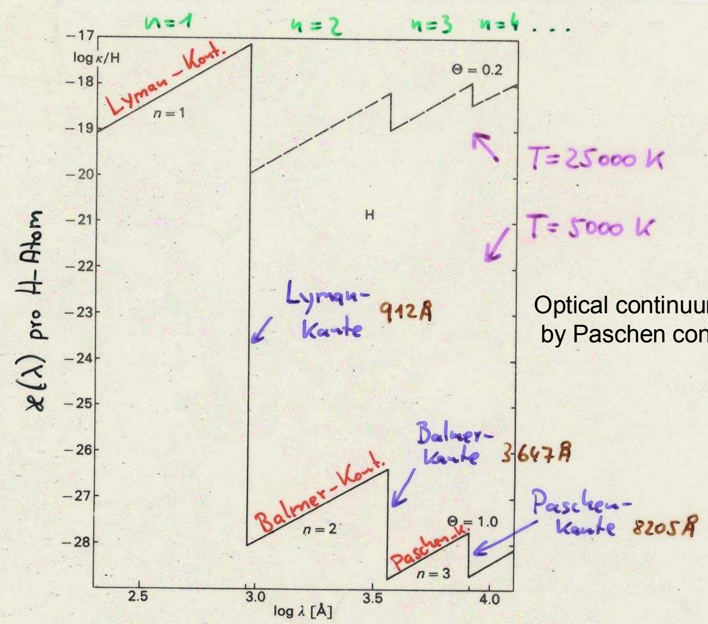
n principal quantum number, Z nuclear charge

Quantum mechanical calculations yield correction factors

$$\sigma(\nu) = \sigma_{\mathrm{th}} (\nu_{\mathrm{th}}/\nu)^3 g_{\mathrm{bf}}(n,\nu) \quad , \quad g_{\mathrm{bf}}(n,\nu) \text{ Gaunt factor}$$

Adding up of bound-free absorptions from all atomic levels:
example hydrogen

$$\kappa_{\mathrm{bf}}^{\mathrm{tot}}(\nu) = \sum_{n=1}^{n_{\max}} \sigma_{\mathrm{bf}}^n(\nu) n_n$$



Optical continuum dominated by Paschen continuum

Fig. 7.3. The hydrogen absorption coefficient κ_λ per hydrogen atom is shown as a function of wavelength for two temperatures $\Theta = 5040/T = 0.2$ and for $\Theta = 5040/T = 1.0$. Higher temperatures lead to higher values of κ_λ in the visual spectral region. (Böhm-Vitense)

The solar continuum spectrum and the H$^-$ ion

H$^-$ ion has one bound state, ionization energy 0.75 eV
Absorption edge near 17000Å,
hence, can potentially contribute to opacity in optical band

$$\text{Sun: } T = 6000\text{K},\ \log n_e = 13.6 \quad \text{Saha equation: } \frac{n_{H^+}}{n_{H^0}} = 10^{-4}, \frac{n_{H^-}}{n_{H^0}} = 10^{-7.5}$$

H almost exclusively neutral, but in the optical Paschen-continuum, i.e. population of H(n=3) decisive:

$$\frac{n_{H^0}(n=3)}{n_{H^0}(n=1)} = \frac{g_3}{g_1} e^{-12.1\text{eV}/kT} = \frac{18}{2} e^{-23.4} = 6 \cdot 10^{-10}$$

$$\frac{n_{H^-}}{n_{H^0}(n=3)} = \frac{n_{H^-}}{n_{H^0}(n=1)} \frac{n_{H^0}(n=1)}{n_{H^0}(n=3)} = \frac{3 \cdot 10^{-8}}{6 \cdot 10^{-10}} = 500$$

Bound-free cross-sections for H$^-$ and H^0 are of similar order

H$^-$ bound-free opacity therefore dominates the visual continuum spectrum of the Sun

Stel

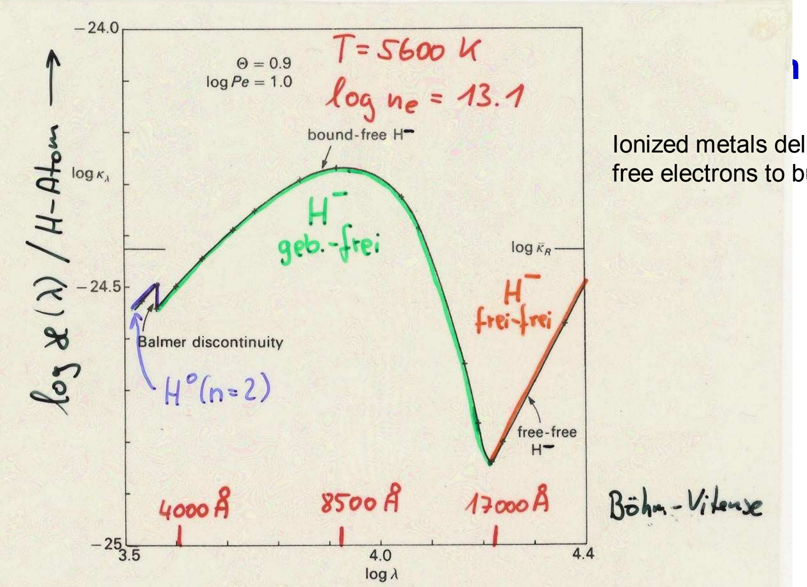


Fig. 7.5. The continuous absorption coefficient per particle is shown for $\Theta = 0.9$, which means $T = 5600$, and $\log P_e = 1.0$. The contributions from the bound-free and free-free continua of H^- are dominant. The small contribution from the Balmer continuum of hydrogen is visible at $\log \lambda = 3.562$. $\bar{\kappa}_R$ is the Rosseland mean absorption coefficient; see section 8.4.

Ionized metals deliver free electrons to build H^-

47

spectrum and the H^- ion

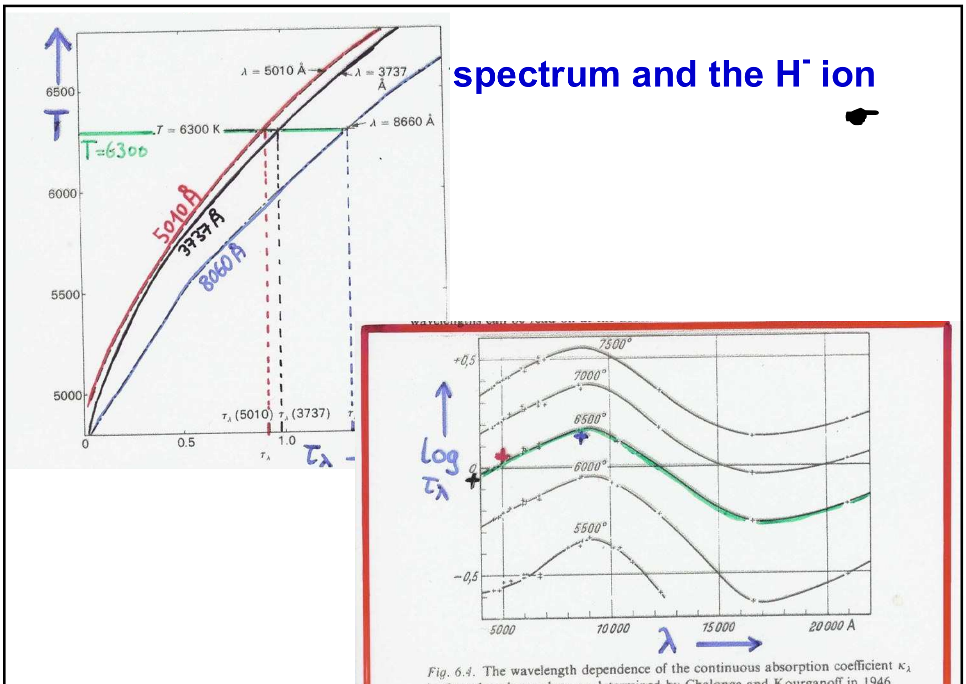
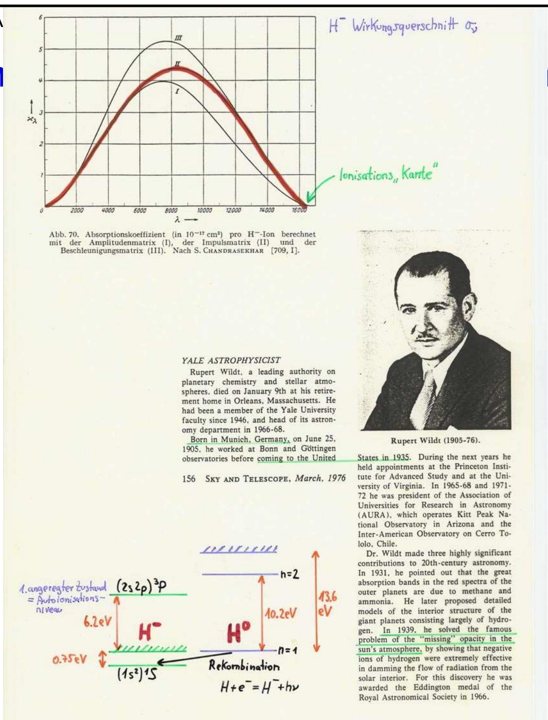


Fig. 6.4. The wavelength dependence of the continuous absorption coefficient κ_λ in the solar photosphere as determined by Chalonge and Kourganoff in 1946. (From Unsöld, 1955, p. 116.)

Th nd the H$^-$ ion



Scattering processes

Thomson scattering at free electrons

Absorption coefficient $\kappa = n_e \sigma_e$ follows from power of harmonic oscillator (σ_e Thomson cross-section)

$$\overline{p} = \left(\frac{e^4 E_0^2}{3m^2c^3} \right) \left[\frac{\nu^4}{\left(\nu_0^2 - \nu^2\right)^2 + \left(\gamma/2\pi\right)^2 \nu^2} \right]$$

free electrons: no resonance frequency, no friction: $\nu_0 = 0; \gamma = 0$

$$\rightarrow \overline{p} = \frac{e^4 E_0^2}{3m^2c^3} \quad , \quad \text{on the other hand we had} \quad \overline{p} = \sigma_e \frac{c}{8\pi} E_0^2$$

$$\boxed{\sigma_e = \frac{8\pi}{3} \frac{e^4}{m^2c^4} = 6.65 \cdot 10^{-25}\ \text{cm}^2}$$

Thomson cross-section is wavelength-independent

Scattering processes

Rayleigh scattering of photons on electrons bound in atoms or molecules

$$\bar{p} = \left(\frac{e^4 E_0^2}{3m^2c^3}\right)\left[\frac{\nu^4}{\left(\nu_0^2-\nu^2\right)^2+\left(\gamma/2\pi\right)^2\nu^2}\right]$$

semi-classical: $\nu \ll \nu_0 = \nu_{lu}$

$$\rightarrow \bar{p} = \frac{e^4 E_0^2}{3m^2c^3}\frac{\nu^4}{\nu_{lu}{}^4} \quad \text{on the other hand we had} \quad \bar{p} = \sigma_\mathrm{R}\frac{c}{8\pi}E_0^2$$

$$\boxed{\sigma_\mathrm{R} = \frac{8\pi}{3}\frac{e^4}{m^2c^4}\frac{\nu^4}{\nu_{lu}{}^4}f_\mathrm{lu} = \sigma_\mathrm{e} f_\mathrm{lu}\frac{\nu^4}{\nu_{lu}{}^4}}$$

(here we have included the oscillator strength as the quantum mechanical correction)

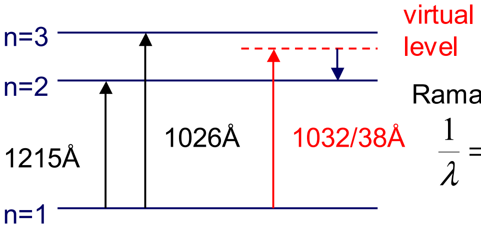
$$\rightarrow \kappa_\mathrm{R}(\nu) = \sum_l n_l \sigma_\mathrm{e} f_\mathrm{lu}\frac{\nu^4}{\nu_{lu}{}^4}$$

Rayleigh scattering on Lyα important for stellar spectral types G and K

Raman scattering

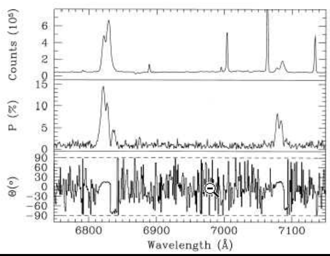
Discovered in symbiotic nova RR Tel

Raman scattering of O VI resonance line (Schmid 1987)



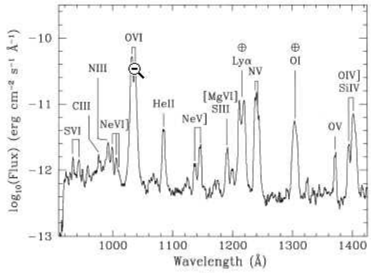
Raman-scattered line 6825/7082Å

$$\frac{1}{\lambda} = \frac{1}{\lambda_\mathrm{OVI}} - \frac{1}{\lambda_{\mathrm{Ly}\,\alpha}}$$



Schmid 1989,

Espey et al. 1995



Two-photon processes

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Maria Goeppert Mayer

1906-1972

Some Important Contributions:

NUCLEAR SHELL MODEL:

Discovery of the magic numbers and their explanation in terms of a nuclear shell model with strong spin-orbit coupling. For this she won the 1963 Nobel Prize in Physics, with J.H.D. Jensen who had independently proposed the strong spin-orbit coupling.

She was the first person to investigate the theoretical basis of nuclear pairing, which plays an important role in the shell model of the atomic nucleus.

OTHER IMPORTANT CONTRIBUTIONS:

Maria Goeppert Mayer was an accomplished physicist from the beginning of her career until the end and she made numerous contributions to the field of physics. She was the first person to investigate the phenomenon of double quantum emission and, a few years later, double beta decay. Mayer and Herzefeld were the first to study the effect of magnetic susceptibility on the refractive index of a gas. Mayer and Sachs pioneered the application of the new idea of a Yukawa potential between neutron and proton to the nuclear two-body system. Mayer was the first person to work out the atomic properties of transuranic elements as well. Mayer's last contribution, with Lawson, was the use of the center of mass and relative coordinates for the calculation of shell model interaction energies.

Some Important Publications:

Free-free absorption and emission

Assumption (also valid in non-LTE case):

Electron velocity distribution in TE, i.e. Maxwell distribution

$$S_\nu^{\mathrm{ff}}(\nu) = \eta_\nu^{\mathrm{ff}}(\nu) / \kappa^{\mathrm{ff}}(\nu) = B_\nu(\nu, T)$$

Free-free processes always in TE

Similar to bound-free process we get:

$$\kappa^{\mathrm{ff}}(\nu) = \sigma_{\mathrm{ff}}(\nu) n_e n_k \left(1 - e^{-h\nu/kT}\right)$$

$$\sigma_{\mathrm{ff}}(\nu) = \frac{16\pi^2}{3\sqrt{3}} \cdot \frac{Z^2 e^6}{hc(2\pi m)^{3/2}} \cdot \frac{1}{\nu^3} \frac{1}{\sqrt{T}} g_{\mathrm{ff}}(n, \nu, T)$$

generalized Kramers formula, with Gauntfaktor from q.m.

- Free-free opacity important at higher energies, because less and less bound-free processes present
- Free-free opacity important at high temperatures

$$\sigma_{\mathrm{ff}} \sim T^{-1/2}, \text{ but } \quad \sigma_{\mathrm{bf}} \sim T^{-3/2} \text{ (Saha), therefore: } \quad \kappa_{\mathrm{ff}}/\kappa_{\mathrm{bf}} \propto T$$

Computation of population numbers

General case, non-LTE: $n_i = n_i(\rho, T, I_\nu)$
In LTE, just $n_i = n_i(\rho, T)$

In LTE completely given by:

- Boltzmann equation (excitation within an ion)
- Saha equation (ionization)

Boltzmann equation

Derivation in textbooks

$$\boxed{\frac{n_i}{n_j} = \frac{g_i}{g_j} e^{-(E_i - E_j)/kT}}$$

g_i statistical weight
E_i excitation energy

Other formulations:

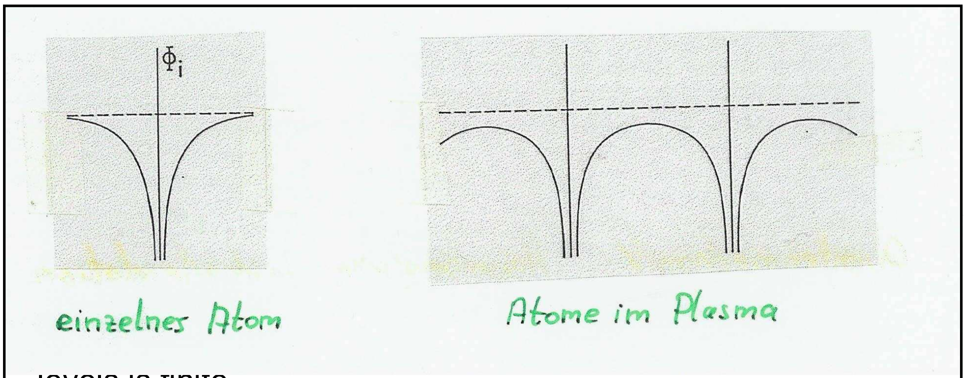
- Related to ground state (E_1=0)

$$\frac{n_i}{n_1} = \frac{g_i}{g_1} e^{-E_i/kT}$$

- Related to total number density N of respective ion

$$\frac{n_i}{\sum n_j} = \frac{n_i}{n_1} \frac{n_1}{\sum n_j} = \frac{n_i}{n_1} \frac{1}{\sum \frac{n_j}{n_1}} = \frac{n_i}{n_1} \frac{g_1}{\sum g_j e^{-E_j/kT}}$$

$$\rightarrow \frac{n_i}{N} = \frac{n_i}{n_1} \frac{g_1}{U(T)}, \quad \text{with partition function } U(T) := \sum g_j e^{-E_j/kT}$$



levels is finite.

Very highly excited levels cannot exist because of interaction with neighbouring particles, radius H atom: $r(n) = a_0 n^2$

At density 10^{15} atoms/cm^3 → mean distance about 10^{-5} cm

$r(n_{max}) = 10^{-5}$ cm → $n_{\max}$ ~43

Levels are "dissolved"; description by concept of occupation probabilities p_i (Mihalas, Hummer, Däppen 1991)

$$g_i \rightarrow g_i p_i \quad \text{with} \quad p_i \rightarrow 0 \quad \text{when} \quad i \rightarrow \infty$$

Hummer-Mihalas occupation probabilities

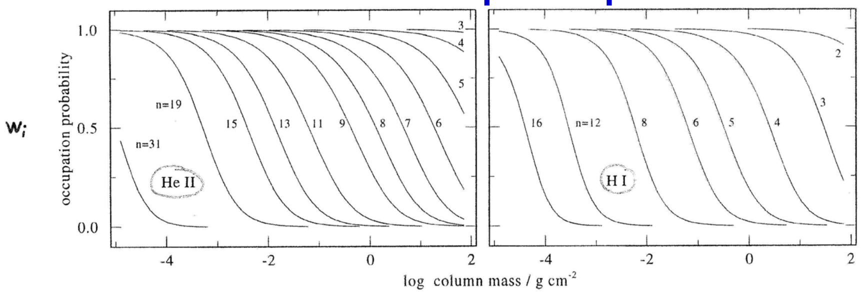
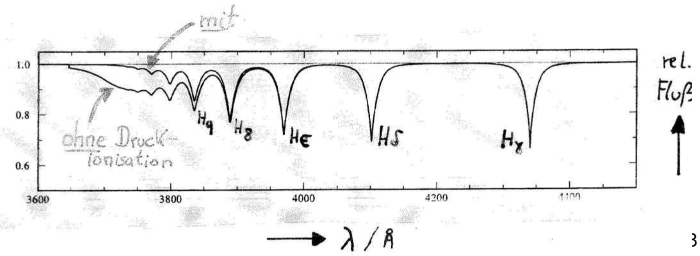


Fig. 2. Occupation probabilities of atomic He II levels (left) and H I levels (right) as a function of depth in the DO model atmosphere with $T_{\rm eff}$=100 000 K, log g =7.5, and H/He=0.1%.



Saha equation

Derivation with Boltzmann formula, but upper state is now a 2-particle state (ion plus free electron)

Energy: $\mathrm{E} = \mathrm{E}_{\mathrm{ion}} + \mathrm{p}^2 / 2\mathrm{m}_{\mathrm{e}}$ p=electron momentum)

Statistical weight: $g = g_{\mathrm{up}} \cdot G(p)$ weight of ion * weight of free electron

Insert into Boltzmann formula

$$\frac{n_{\mathrm{up}}(p)}{n_{\mathrm{low}}} = \frac{g_{\mathrm{up}} G(p)}{g_{\mathrm{low}}} e^{-(E_{\mathrm{ion}} + p^2/2m_{\mathrm{e}} - E_{\mathrm{low}})/kT}$$

Summarize over all final states
By integration over p

$$\rightarrow \frac{n_{\mathrm{up}}}{n_{\mathrm{low}}} = \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} e^{-(E_{\mathrm{up}} - E_{\mathrm{low}})/kT} \int_0^\infty G(p) e^{-p^2/2m_{\mathrm{e}}kT} dp$$

Statistical weight of free electron =number of available states in interval [p,p+dp] (Pauli principle):

$$G(p)dp = 2 \frac{d\Omega(p)}{h^3} \quad \begin{matrix} \\ \text{2 spins} \end{matrix} \begin{matrix} \text{phase space volume} \\ \text{phase space cell} \end{matrix}$$

$$d\Omega(p) = dxdydz \cdot dp_x dp_y dp_z = dV \cdot 4\pi p^2 dp = 1/n_{\mathrm{e}} \cdot 4\pi p^2 dp \rightarrow G(p) = 8\pi p^2 / h^3 n_{\mathrm{e}}$$

Saha equation

Insertion into Boltzmann formula gives:

$$\frac{n_{\mathrm{up}}}{n_{\mathrm{low}}} = \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} e^{-(E_{\mathrm{up}} - E_{\mathrm{low}})/kT} \int_0^\infty \frac{8\pi}{h^3 n_{\mathrm{e}}} p^2 e^{-p^2/2mkT_{\mathrm{e}}} dp \quad \text{with} \quad x = p / \sqrt{2m_{\mathrm{e}}kT}$$

$$= \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} e^{-(E_{\mathrm{up}} - E_{\mathrm{low}})/kT} \frac{8\pi}{h^3 n_{\mathrm{e}}} (2m_{\mathrm{e}}kT)^{3/2} \int_0^\infty x^2 e^{-x^2} dx$$

$$= \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} e^{-(E_{\mathrm{up}} - E_{\mathrm{low}})/kT} \frac{8\pi}{h^3 n_{\mathrm{e}}} (2m_{\mathrm{e}}kT)^{3/2} \frac{\sqrt{\pi}}{4}$$

$$\boxed{\frac{n_{\mathrm{up}}}{n_{\mathrm{low}}} = \frac{2}{n_{\mathrm{e}}} \left(\frac{2\pi m_{\mathrm{e}} kT}{h^3} \right)^{3/2} \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} e^{-(E_{\mathrm{up}} - E_{\mathrm{low}})/kT}}$$

Saha equation for two levels in adjacent ionization stages

Alternative: $\dfrac{n_{\mathrm{up}} n_{\mathrm{e}}}{n_{\mathrm{low}}} = f(T) = \dfrac{T^{3/2}}{C} \dfrac{g_{\mathrm{up}}}{g_{\mathrm{low}}} e^{-(E_{\mathrm{up}} - E_{\mathrm{low}})/kT}$ $\quad C = 2.07 \cdot 10^{-16}\ \mathrm{K}^{3/2}\ \mathrm{cm}^3$

Example: hydrogen

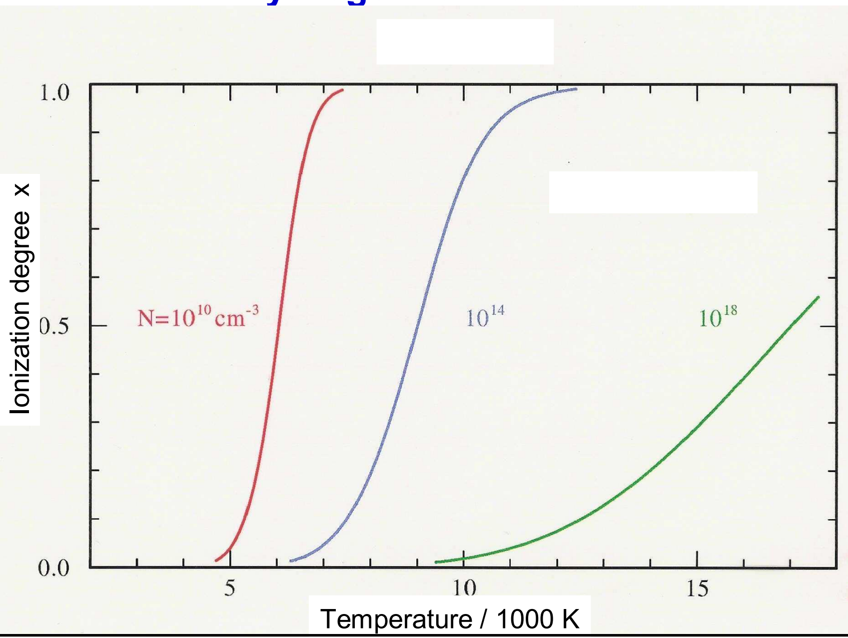
Model atom with only one bound state:

$$n_{\text{low}} = n_{\text{I}} = n(\text{H I ground state}) \qquad g_{\text{I}} = 2$$
$$n_{\text{up}} = n_{\text{II}} = n(\text{H II}) \qquad g_{\text{II}} = 1$$
$$\frac{n_e n_{\text{II}}}{n_{\text{I}}} = \frac{T^{3/2}}{C}\frac{1}{2}e^{-1.58\cdot 10^5\,\text{K}/T} = f(T)$$
pure hydrogen: $n_e = n_{\text{II}}$, $N = n_{\text{I}} + n_{II}$

ionization degree: $x = \frac{n_e}{N} = \frac{n_{\text{II}}}{N}$

$$\Rightarrow \frac{x^2 N}{1-x} = f(T) \Rightarrow x^2 + \frac{f(T)}{N}x - \frac{f(T)}{N} = 0$$
$$\boxed{\Rightarrow x = -\frac{f(T)}{2N} + \sqrt{\left(\frac{f(T)}{2N}\right)^2 + \frac{f(T)}{N}}}$$
$$\Rightarrow x = x(T, N)$$

Hydrogen ionization



More complex model atoms

$j=1,\dots,J$ ionization stages

$i=1,\dots,I(j)$ levels per ionization stage j

Saha equation for ground states of ionization stages j and $j+1$:

$$n_{1\mathrm{j}} = n_{1\mathrm{j+1}} n_\mathrm{e} \frac{1}{2}\left(\frac{h^3}{2\pi m_\mathrm{e} kT}\right)^{3/2} \frac{g_{1\mathrm{j}}}{g_{1\mathrm{j+1}}} e^{E^\mathrm{j}_\mathrm{Ion}/kT}$$

With Boltzmann formula we get occupation number of i-th level:

$$n_\mathrm{ij} = \frac{n_\mathrm{ij}}{n_{1\mathrm{j}}} n_{1\mathrm{j}} = \frac{g_\mathrm{ij}}{g_{1\mathrm{j}}} e^{-E^\mathrm{j}_\mathrm{i}/kT} n_{1\mathrm{j+1}} n_\mathrm{e} C_1 T^{-3/2} \frac{g_{1\mathrm{j}}}{g_{1\mathrm{j+1}}} e^{E^\mathrm{j}_\mathrm{Ion}/kT}$$

$$\boxed{\Rightarrow n_\mathrm{ij} = \frac{g_\mathrm{ij}}{g_{1\mathrm{j+1}}} n_{1\mathrm{j+1}} n_\mathrm{e} C_1 T^{-3/2} e^{(E^\mathrm{j}_\mathrm{Ion} - E^\mathrm{j}_\mathrm{i})/kT}}$$

More complex model atoms

Related to total number of particles in ionization stage j+1

$$\frac{n_\mathrm{ij+1}}{N_\mathrm{j+1}} = \frac{n_\mathrm{ij+1}}{n_{1\mathrm{j+1}}} \frac{g_{1\mathrm{j+1}}}{U_\mathrm{j+1}} \underset{\mathrm{i=1}}{=} \frac{n_{1\mathrm{j+1}}}{N_\mathrm{j+1}} = 1 \cdot \frac{g_{1\mathrm{j+1}}}{U_\mathrm{j+1}} \to n_{1\mathrm{j+1}} = \frac{g_{1\mathrm{j+1}}}{U_\mathrm{j+1}} N_\mathrm{j+1}$$

$$\Rightarrow n_\mathrm{ij} = \frac{g_\mathrm{ij}}{g_{1\mathrm{j+1}}} \frac{g_{1\mathrm{j+1}}}{U_\mathrm{j+1}} N_\mathrm{j+1} n_\mathrm{e} C_1 T^{-3/2} e^{(E^\mathrm{j}_\mathrm{Ion} - E^\mathrm{j}_\mathrm{i})/kT} \Rightarrow \boxed{n_\mathrm{ij} = \frac{g_\mathrm{ij}}{U_\mathrm{j+1}} N_\mathrm{j+1} n_\mathrm{e} C_1 T^{-3/2} e^{(E^\mathrm{j}_\mathrm{Ion} - E^\mathrm{j}_\mathrm{i})/kT}}$$

N_j/N_{j+1}

$$N_\mathrm{j} = \sum_\mathrm{i} n_\mathrm{ij} = \sum_\mathrm{i} \frac{g_\mathrm{ij}}{U_\mathrm{j+1}} N_\mathrm{j+1} n_\mathrm{e} C_1 T^{-3/2} e^{(E^\mathrm{j}_\mathrm{Ion} - E^\mathrm{j}_\mathrm{i})/kT}$$

$$= \frac{N_\mathrm{j+1}}{U_\mathrm{j+1}} n_\mathrm{e} C_1 T^{-3/2} e^{E^\mathrm{j}_\mathrm{Ion}/kT} \sum_\mathrm{i} g_\mathrm{ij} e^{-E^\mathrm{j}_\mathrm{i}/kT} = \frac{N_\mathrm{j+1}}{U_\mathrm{j+1}} n_\mathrm{e} C_1 T^{-3/2} e^{E^\mathrm{j}_\mathrm{Ion}/kT} U_\mathrm{j}$$

$$\boxed{\frac{N_\mathrm{j}}{N_\mathrm{j+1}} = \frac{U_\mathrm{j}}{U_\mathrm{j+1}} n_\mathrm{e} C_1 T^{-3/2} e^{E^\mathrm{j}_\mathrm{Ion}/kT} = n_\mathrm{e} \Phi_\mathrm{j}(T)}$$

Ionization fraction

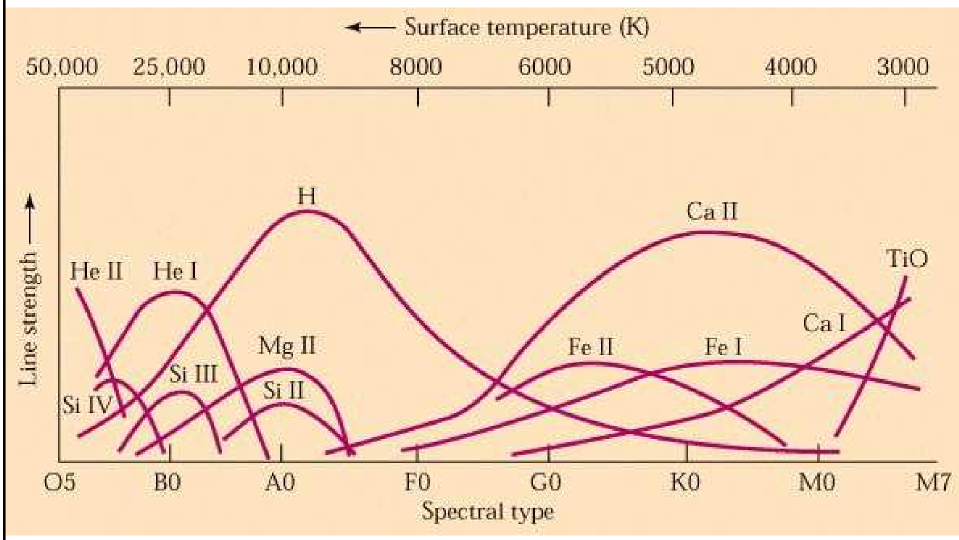
$$\frac{N_j}{N_J} = \frac{N_j}{N_{j+1}} \cdot \frac{N_{j+1}}{N_{j+2}} \cdot \ldots \cdot \frac{N_{J-1}}{N_J}$$

$$N = \sum_{j=1}^{J} N_j = N_J \sum_{j=1}^{J} \frac{N_j}{N_J} = N_J \left[1 + \frac{N_{J-1}}{N_J} + \frac{N_{J-1}}{N_J} \cdot \frac{N_{J-2}}{N_{J-1}} + \ldots + \frac{N_{J-1}}{N_J} \cdots \frac{N_1}{N_2} \right]$$

$$\frac{N_j}{N} = \frac{N_j}{N_J} \frac{N_J}{N} = \frac{\frac{N_j}{N_{j+1}} \cdot \frac{N_{j+1}}{N_{j+2}} \cdot \ldots \cdot \frac{N_{J-1}}{N_J}}{1 + \frac{N_{J-1}}{N_J} + \frac{N_{J-1}}{N_J} \cdot \frac{N_{J-2}}{N_{J-1}} + \ldots + \frac{N_{J-1}}{N_J} \cdots \frac{N_1}{N_2}}$$

$$\boxed{\frac{N_j}{N} = \frac{\prod_{k=j}^{J-1} n_e \Phi_k(T)}{1 + \sum_{m=1}^{J} \prod_{k=m}^{J-1} n_e \Phi_k(T)}}$$

Ionization fractions



Summary: Emission and Absorption

- Line absorption and emission coefficients (bound-bound)

$$\kappa_{lu}(\nu) = \frac{\pi e^2}{mc} f_{lu} \left(n_{low} - \frac{g_{low}}{g_{up}} n_{up} \right) \varphi(\nu) \qquad \eta_{lu}(\nu) = \frac{2h\nu_0^3}{c^2} \frac{\pi e^2}{mc} f_{lu} \frac{g_{low}}{g_{up}} n_{up} \varphi(\nu)$$

$\varphi(\nu) =$ profile function, e.g., Voigtprofile $V(a, \mathrm{v}) = \dfrac{1}{\Delta\nu_D \sqrt{\pi}} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-y^2}}{(\mathrm{v} - y)^2 + a^2} dy$

- Continuum (bound-free)

$$\kappa(\nu) = \sigma_\nu \left[n_{low} - n_{up} \left(\frac{n_{up}}{n_{low}} \right)^* e^{-h\nu/kT} \right] \qquad \eta_\nu(\nu) = \frac{2h\nu}{c^3} \sigma_\nu n_{up} \left(\frac{n_{up}}{n_{low}} \right)^* e^{-h\nu/kT}$$

- Continuum (free-free), always in LTE

$$\kappa^{ff}(\nu) = \sigma_{ff}(\nu) n_e n_k \left(1 - e^{-h\nu/kT} \right) \qquad \eta^{ff}(\nu) = \kappa^{ff}(\nu) n_e n_k B_\nu(\nu, T)$$

- Scattering (Compton, on free electrons) $\kappa = n_e \sigma_e \qquad \eta_\nu(\nu) = n_e \sigma_e J_\nu$

Total opacity and emissivity add up all contributions, then source function $S_\nu = \eta_\nu / \kappa(\nu)$

Excitation and ionization in LTE

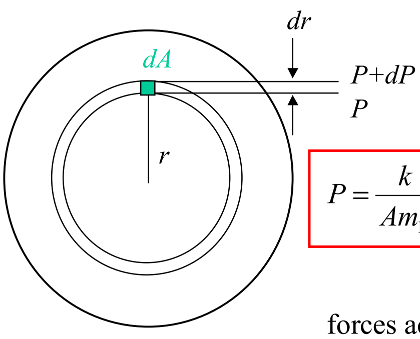
$$\frac{n_{low}}{n_{up}} = \frac{g_{low}}{g_{up}} e^{-(E_{low}-E_{up})/kT}$$ Boltzmann

$$\frac{n_{\mathrm{up}}}{n_{\mathrm{low}}} = \frac{2}{n_{\mathrm{e}}} \left(\frac{2\pi m_{\mathrm{e}} kT}{h^3} \right)^{3/2} \frac{g_{\mathrm{up}}}{g_{\mathrm{low}}} e^{-(E_{\mathrm{up}}-E_{\mathrm{low}})/kT}$$ Saha

Hydrostatic Equilibrium

Particle conservation

Ideal gas



$$\boxed{P = \frac{k}{Am_{\mathrm{H}}} \rho \cdot T}$$

$P = pressure$
$\rho =$ mass density
$A =$ atomic weight

forces acting on volume element:

$dV = dAdr \quad dm = \rho dV$

$$dF_g = -\frac{GM_r dm}{r^2} = -\frac{GM_r \rho}{r^2} dAdr$$

buoyancy:

$dF_P = -dPdA$ (pressure difference * area)

Ideal gas

In stellar atmospheres:

$M_r = M_*$ mass of atmosphere negligible

$r = R_*$ thickness of atmosphere << stellar radius

$$\rightarrow dF_g = -\frac{GM_*\rho}{R_*^2} dAdr = g\rho dAdr$$

with $g := \frac{GM_*}{R_*^2}$ surface gravity

usually written as $\log(g/\mathrm{cm\ s^{-2}})$

log *g* is besides T_{eff} the 2nd fundamental parameter of static stellar atmospheres

Type	log *g*
Main sequence star	4.0 .... 4.5
Sun	4.44
Supergiants	0 1
White dwarfs	~8
Neutron stars	~15
Earth	3.0

Hydrostatic equilibrium, ideal gas

buoyancy = gravitational force:

$$dF_p + dF_g = 0$$

$$-dPdA - g\rho dAdr = 0$$

$$\boxed{\frac{dP}{dr} = -g\rho(r)}$$

eliminate $\rho(r)$ with ideal gas equation: $\frac{dP}{dr} = -g\frac{A(r)m_\mathrm{H}}{kT(r)} P(r)$

example:

$T(r) = T = \mathrm{const}$, $A(r) = A = \mathrm{const}$ (i.e., no ionization or dissociation)

$$\frac{dP}{dr} = -g\frac{Am_\mathrm{H}}{kT} P(r) \Rightarrow \frac{1}{P}\frac{dP}{dr} = -g\frac{Am_\mathrm{H}}{kT}$$

solution:

$$\mathrm{P(r)} = \mathrm{P(r_0)} e^{-(r-r_0)gAm_\mathrm{H}/kT}$$

$$\boxed{\mathrm{P(r)} = \mathrm{P(r_0)} e^{-(r-r_0)/H} \qquad H := \frac{kT}{gAm_\mathrm{H}} \text{ pressure scale height}}$$

Atmospheric pressure scale heights

Earth:

$$\left.\begin{array}{r} A \approx 28\,(\mathrm{N_2}) \\ T \approx 300\,\mathrm{K} \\ \log g = 3 \end{array}\right\} H = 9\,\mathrm{km} \qquad H = \frac{kT}{gAm_\mathrm{H}}$$

Sun:

$$\left.\begin{array}{r} A = 1\,(\mathrm{H}) \\ T \approx 6000\,\mathrm{K} \\ \log g = 4.44 \end{array}\right\} H = 180\,\mathrm{km}$$

White dwarf:

$$\left.\begin{array}{r} A = 0.5\,(\mathrm{H}^+ + n_e) \\ T = 15000\,\mathrm{K} \\ \log g = 8 \end{array}\right\} H = 0.25\,\mathrm{km}$$

Neutron star:

$$\left.\begin{array}{r} A = 0.5\,(\mathrm{H}^+ + n_e) \\ T = 10^6\,\mathrm{K} \\ \log g = 15 \end{array}\right\} H = 1.6\,\mathrm{mm}\ !$$

Effect of radiation pressure

2nd moment of intensity $P_R(\nu) = \frac{4\pi}{c} K_\nu$

1st moment of transfer equation (plane-parallel case)

$$\frac{dK_\nu}{d\tau(\nu)} = H_\nu$$

$$\frac{dP_R}{d\tau(\nu)} = \frac{4\pi}{c} H_\nu \quad \text{with} \quad d\tau(\nu) = \kappa(\nu) dr$$

$$\frac{dP_R}{dr} = \frac{4\pi}{c} \kappa(\nu) H_\nu$$

integration over frequencies:

$$\boxed{\frac{dP_R}{dr} = \frac{4\pi}{c} \int_0^\infty \kappa(\nu) H_\nu d\nu}$$

Effect of radiation pressure

Extended hydrostatic equation

$$\frac{dP}{dr} = g\rho(r) - \frac{dP_R}{dr} = g\rho(r) - \frac{4\pi}{c}\int_0^\infty \kappa(\nu) H_\nu d\nu$$

$$= g_{\rm eff}(r)\rho(r)$$

definition: effective gravity

$$g_{\rm eff}(r) := g - \frac{4\pi}{c}\frac{1}{\rho(r)}\int_0^\infty \kappa(\nu) H_\nu d\nu = g - g_{\rm rad} \quad \text{(depth dependent!)}$$

In the outer layers of many stars:

$$g_{\rm eff} < 0 \quad \text{i.e.} \quad g_{\rm rad} = \frac{4\pi}{c}\frac{1}{\rho(r)}\int_0^\infty \kappa(\nu) H_\nu d\nu > g$$

Atmosphere is no longer static, hydrodynamical equation

Expanding stellar atmospheres, radiation-driven winds

The Eddington limit

Estimate radiative acceleration

Consider only (Thomson) electron scattering as opacity

$\sigma(\nu) = \sigma_{\rm e}$ (Thomson cross-section)

$q =$ number of free electrons per atomic mass unit

Pure hydrogen atmosphere, completely ionized

$q = 1$

Pure helium atmosphere, completely ionized

$q = 2/4 = 0.5$

$$g_{\rm rad}^{\rm e} = \frac{4\pi}{c}\frac{1}{n_{\rm e} m_H / q}\int_0^\infty \sigma_{\rm e} n_{\rm e} H_\nu d\nu = \frac{4\pi}{c}\frac{q}{m_H}\int_0^\infty \sigma_{\rm e} H_\nu d\nu = \frac{4\pi}{c}\frac{q\sigma_{\rm e}}{m_H} H$$

Flux conservation: $H = \frac{\sigma}{4\pi} T_{\rm eff}^4$

$$\Gamma_{\rm e} = \frac{g_{\rm rad}^{\rm e}}{g} = \frac{4\pi}{c}\frac{q\sigma_{\rm e}}{\rm m_H}\frac{\sigma}{4\pi}T_{\rm eff}^4 \Big/ G\frac{M}{R^2} = \frac{1}{c}\frac{q\sigma_{\rm e}}{\rm m_H}\frac{1}{4\pi G}\frac{4\pi\sigma R^2 T_{\rm eff}^4}{M}$$

$$\boxed{\Gamma_{\rm e} = \frac{q\sigma_{\rm e}}{4\pi c {\rm m_H} G}\frac{L}{M} = 10^{-4.51} q \frac{L/L_\odot}{M/M_\odot}}$$

The Eddington limit

Consequence: for given stellar mass there exists a maximum luminosity. No stable stars exist above this luminosity limit.

$$L_{\max}/L_{\odot} = 10^{-4.51} \cdot 1/q \cdot M/M_{\odot}$$

Sun: $\Gamma_e << 1$

Main sequence stars (central H-burning)

Mass luminosity relation: $L/L_{\odot} \approx (M/M_{\odot})^3 \rightarrow M_{\max} = 180 M_{\odot}$

Gives a mass limit for main sequence stars

Eddington limit written with effective temperature

and gravity $\Gamma_e = 10^{-15.12} q T_{\text{eff}}^4 / g = 1$

$$-15.12 + \log q + 4 \log T_{\text{eff}} - \log g = 0$$

Straight line in ($\log T_{eff}$, $\log g$)-diagram

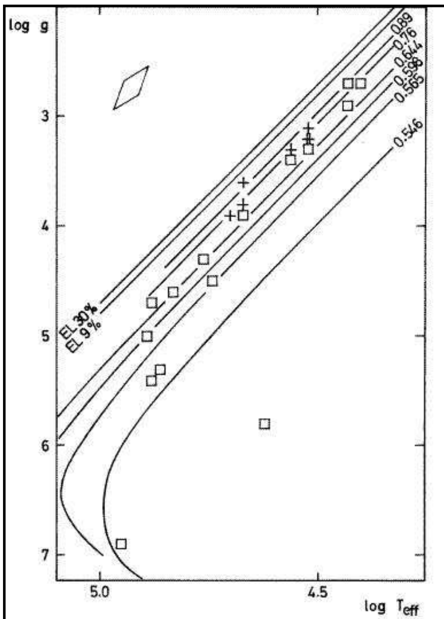


Fig. 3. The log g-log T_{eff} diagram. The two lines labeled EL are Eddington limits for photospheric He abundances of 30% and 9%. We have plotted 6 theoretical post-AGB evolutionary tracks, which are labeled with the corresponding value of the stellar mass, in solar masses. Plus signs and open squares indicate CSPN that show, respectively, He II λ4686 in emission and in absorption. A typical error box can be seen in the upper left corner of the figure

The Eddington limit

Positions of analyzed central stars of planetary nebulae

and

theoretical stellar evolutionary tracks (mass labeled in solar masses)

Computation of electron density

At a given temperature, the hydrostatic equation gives the gas pressure at any depth, or the total particle density N:

$$P_{\text{gas}} = NkT$$

$$N = N_{\text{atoms}} + N_{\text{ions}} + n_{\text{e}} = N_{\text{N}} + n_{\text{e}}$$ N_{N} massive particle density

The Saha equation yields for given (n_e, T) the ion- and atomic densities N_{N}.

The Boltzmann equation then yields for given (N_{N}, T) the population densities of all atomic levels: n_i.

Now, how to get n_e?

We have $\boldsymbol{k}$ different species with abundances α_k

Particle density of species $\boldsymbol{k}$:

$$N_k = \alpha_k N_{\text{N}} = \alpha_k \left(N - n_{\text{e}} \right) \quad \text{, and it is} \quad \sum_{\text{k}=1}^{\text{K}} N_k = N_{\text{N}}$$

Charge conservation

Stellar atmosphere is electrically neutral

Charge conservation electron density=ion density * charge

$$n_{\text{e}} = \sum_{k=1}^{K} \sum_{j=1}^{jk} j \cdot N_{jk} \quad , \; N_{jk} = \text{density of j-th ionization stage of species k}$$

Combine with Saha equation (LTE)
by the use of ionization fractions:

$$f_{jk} = \frac{N_{jk}}{N_k} = \frac{\prod_{l=j}^{jk\text{-}1} n_{\text{e}} \Phi_{lk}(T)}{1 + \sum_{m=1}^{jk} \prod_{l=m}^{jk\text{-}1} n_{\text{e}} \Phi_{lk}(T)}$$

We write the charge conservation as

$$n_{\text{e}} = \sum_{k=1}^{K} \sum_{j=1}^{jk} j \cdot N_k f_{jk}(n_{\text{e}}, T) = \sum_{k=1}^{K} \alpha_k (N - n_{\text{e}}) \sum_{j=1}^{jk} j \cdot f_{jk}(n_{\text{e}}, T)$$

$$n_{\text{e}} = (N - n_{\text{e}}) \sum_{k=1}^{K} \alpha_k \sum_{j=1}^{jk} j \cdot f_{jk}(n_{\text{e}}, T) = F(n_{\text{e}})$$

Non-linear equation, iterative solution, i.e., determine zeros of

$$\boxed{F(n_{\text{e}}) - n_{\text{e}} = 0}$$

use Newton-Raphson, converges after 2-4 iterations; yields n_e and f_{ij}, and with Boltzmann all level populations

Summary: Hydrostatic Equilibrium

Summary: Hydrostatic Equilibrium

Hydrostatic equation including radiation pressure

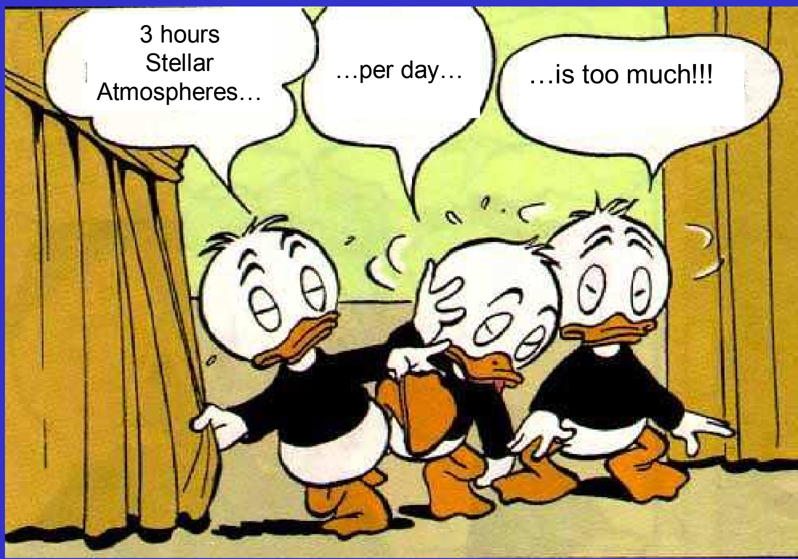
$$\frac{dP}{dr} = g\rho(r) - \frac{dP_R}{dr} = g\rho(r) - \frac{4\pi}{c}\int_0^\infty \kappa(\nu)H_\nu d\nu$$

Photon pressure: Eddington Limit

Hydrostatic equation $\rightarrow N$

Combined charge equation + ionization fraction $\rightarrow n_e$

$\rightarrow$ Population numbers n_{ijk} (LTE) with Saha and Boltzmann equations


3 hours
Stellar
Atmospheres…
…per day…
…is too much!!!

Radiative Equilibrium

Energy conservation

Radiative Equilibrium

Assumption:

Energy conservation, i.e., no nuclear energy sources
Counter-example: radioactive decay of $Ni^{56} \rightarrow Co^{56} \rightarrow Fe^{56}$ in supernova atmospheres

Energy transfer predominantly by radiation

Other possibilities:

Convection e.g., H convection zone in outer solar layer

Heat conduction e.g., solar corona or interior of white dwarfs

Radiative equilibrium means, that we have at each location:

Radiation energy absorbed / sec
=
Radiation energy emitted / sec

integrated over all frequencies and angles

Radiative Equilibrium

Absorption per cm² and second: $\oint_{4\pi} d\omega \int_0^\infty d\nu\, \kappa(\nu) I_\nu$

Emission per cm² and second: $\oint_{4\pi} d\omega \int_0^\infty d\nu\, \eta(\nu)$

Assumption: isotropic opacities and emissivities

Integration over dω then yields

$$\int_0^\infty d\nu\, \kappa(\nu) J_\nu = \int_0^\infty d\nu\, \eta(\nu) \quad \boxed{\Rightarrow \int_0^\infty \kappa(\nu)(J_\nu - S_\nu)\, d\nu = 0}$$

Constraint equation in addition to the radiative transfer equation; fixes temperature stratification ***T(r)***

Conservation of flux

Alternative formulation of energy equation

In plane-parallel geometry: 0-th moment of transfer equation

$$\frac{dH_\nu}{dt} = \kappa(J_\nu - S_\nu)$$

Integration over frequency, exchange integration and differentiation:

$$\frac{d}{dt}\int_0^\infty H_\nu d\nu = \int_0^\infty \kappa\left(J_\nu - S_\nu\right) d\nu = 0 \qquad \text{because of radiative equilibrium}$$

$$\Rightarrow H = \int_0^\infty H_\nu d\nu = \text{const} = \frac{\sigma}{4\pi} T_{\text{eff}}^4 \qquad \text{for all depths. Alternatively written:}$$

$$\int_0^\infty H_\nu d\nu = \frac{\sigma}{4\pi} T_{\text{eff}}^4 = \int_0^\infty \frac{dK_\nu}{d\tau} d\nu \qquad \text{(1st moment of transfer equation)}$$

$$\boxed{\Rightarrow \int_0^\infty \frac{d(f_\nu J_\nu)}{d\tau} d\nu = \frac{\sigma}{4\pi} T_{\text{eff}}^4} \qquad \text{(definiton of Eddington factor)}$$

Which formulation is good or better?

I Radiative equilibrium: local, integral form of energy equation

II Conservation of flux: non-local (gradient), differential form of radiative equilibrium

I / II numerically better behaviour in **small / large** depths

Very useful is a linear combination of both formulations:

$$A \cdot \left[\int_0^\infty \kappa (J_\nu - S_\nu) d\nu \right] + B \cdot \left[\int_0^\infty \frac{d(f_\nu J_\nu)}{d\tau} d\nu - H \right] = 0$$

A,B are coefficients, providing a smooth transition between formulations I and II.

Stellar Atmospheres: Radiative Equilibrium

Flux conservation in spherically symmetric geometry

0-th moment of transfer equation:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 H_\nu \right) = \kappa \left(S_\nu - J_\nu \right)$$

$$\Rightarrow \frac{\partial}{\partial r} \left(r^2 \int_0^\infty H_\nu d\nu \right) = r^2 \int_0^\infty \kappa (S_\nu - J_\nu) d\nu = 0$$

$$r^2 \int_0^\infty H_\nu d\nu = \text{const} = \frac{1}{16\pi^2} L \quad \text{because} \quad L = 16\pi^2 R^2 H$$

Another alternative, if T de-couples from radiation field

Thermal balance of electrons

$$Q^H - Q^C = 0$$

$$Q_{\text{ff}}^H = 4\pi\, n_e \sum_j N_j \int_0^\infty \alpha_{\text{ff},j}(\nu,T) J_\nu d\nu$$

$$Q_{\text{ff}}^C = 4\pi\, n_e \sum_j N_j \int_0^\infty \alpha_{\text{ff},j}(\nu,T) \left(J_\nu + \frac{2h\nu^3}{c^2} \right) e^{-h\nu/kT} d\nu$$

$$Q_{\text{bf}}^H = 4\pi \sum_{l,k} n_l \int_0^\infty \alpha_{\text{bf},lk}(\nu) J_\nu \left(1 - \frac{\nu_{lk}}{\nu} \right) d\nu$$

$$Q_{\text{bf}}^C = 4\pi \sum_{l,k} n_k \int_0^\infty \alpha_{\text{bf},lk}(\nu) J_\nu \left(1 - \frac{\nu_{lk}}{\nu} \right) \left(J_\nu + \frac{2h\nu^3}{c^2} \right) e^{-h\nu/kT} d\nu$$

$$Q_c^H = n_e \sum_{l,m} n_m q_{lm}(T) h\nu_{lm}$$

$$Q_c^C = n_e \sum_{l,m} n_l q_{lm}(T) h\nu_{lm}$$

The gray atmosphere

Simple but insightful problem to solve the transfer equation together with the constraint equation for radiative equilibrium

Gray atmosphere: $\kappa_\nu = \overline{\kappa}$

Moments of transfer equation

$$(I)\quad \frac{dH_\nu}{d\tau} = J_\nu - S_\nu \qquad (II)\quad \frac{dK_\nu}{d\tau} = H_\nu \quad \text{with} \quad \tau = \overline{\kappa} dt$$

Integration over frequency

$$(I)\quad \frac{dH}{d\tau} = J - S \qquad (II)\quad \frac{dK}{d\tau} = H$$

Radiative equilibrium $\int \overline{\kappa}(J_\nu - S_\nu) d\nu = \overline{\kappa} \int (J_\nu - S_\nu) d\nu = J - S = 0$

$\Rightarrow (I)\quad J = S$

and because of conservation of flux $\dfrac{dH}{d\tau} = 0$

$\Rightarrow (II)\quad \dfrac{d^2K}{d\tau^2} = 0 \Rightarrow K = c_1\tau + c_2$ from (II) follows $c_1 = \dfrac{dK}{d\tau} = H,\quad c_2$ see below

The gray atmosphere

Relations *(I)* und *(II)* represent two equations for three quantities *S,J,K* with pre-chosen *H* (resp. T_{eff})

Closure equation: Eddington approximation

$$K = 1/3J \rightarrow S = J = 3K = 3H\tau + 3c_2 \quad (III)$$

Source function is linear in τ

Temperature stratification?

In LTE:

$$S(\tau) = B(T(\tau)) = \frac{\sigma}{\pi} T^4$$

insert into (III): $\frac{\sigma}{\pi} T^4 = 3H\tau + 3c_2$

with $H = \frac{\sigma}{4\pi} T_{\mathrm{eff}}^4$ we get:

$\frac{\sigma}{\pi} T^4(\tau) = \frac{3}{4\pi} \sigma T_{\mathrm{eff}}^4 \tau + 3c_2 \quad (IV)$ $\quad c_2$ is now determined from boundary condition (τ=0)

Gray atmosphere: Outer boundary condition

Emergent flux:

$$H(0) = \frac{1}{2}\int_0^\infty S(\tau')E_2(\tau')d\tau' \quad \text{with } S \text{ from } (III)$$

$$= \frac{1}{2}\int_0^\infty (3H\tau' + 3c_2)E_2(\tau')d\tau'$$

$$= \frac{3}{2}\left[H\int_0^\infty \tau' E_2(\tau')d\tau' + c_2\int_0^\infty E_2(\tau')d\tau'\right]$$

with $\int_0^\infty t^l E_n(t)dt = \frac{l!}{l+n}$ and $E_2(t) = \frac{1}{2-1}\left[e^{-t} - tE_1(t)\right]$

$$H(0) = \frac{3}{2}\left[\frac{1}{3}H + \frac{1}{2}c_2\right] \rightarrow c_2 = \frac{2}{3}H$$

from (IV): $\Rightarrow \boxed{T^4 = \frac{3}{4} T_{\mathrm{eff}}^4 \left(\tau + \frac{2}{3}\right)}$, $S = 3H\left(\tau + \frac{2}{3}\right)$ (from III)

Avoiding Eddington approximation

Ansatz: $J(\tau) = 3H(\tau + q(\tau))$ generalization of (III)

$q(\tau)$ = Hopf function

$$J(\tau) = \frac{3}{4}\frac{\sigma}{\pi} T_{\text{eff}}^4 (\tau + q(\tau))$$

Insert into Schwarzschild equation:

$J(\tau) = \Lambda S = \Lambda J$ integral equation for J

$$\Rightarrow \tau + q(\tau) = \frac{1}{2}\int_0^\infty (\tau' + q(\tau')) E_1(|\tau' - \tau|) d\tau'$$ (*) integral equation for q, see below

Approximate solution for J by iteration ("Lambda iteration")

$J^{(1)} = 3H(\tau + 2/3)$ i.e., start with Eddington approximation

$$J^{(2)} = \Lambda J^{(1)} = \Lambda(3H(\tau + 2/3)) = 3H\left(\tau + \frac{2}{3} - \frac{1}{3}E_2(\tau) + \frac{1}{2}E_3(\tau)\right)$$

(was result for linear S)

At the surface $\tau = 0,\, E_2(0) = 1,\, E_3(0) = \frac{1}{2}$

$$J^{(2)} = 3H\left(\tau + \frac{2}{3} - \frac{1}{3} + \frac{1}{4}\right) = 3H\left(\tau + 0.58\bar{3}\right)$$

exact: q(0)=0.577….

At inner boundary $\tau = \infty,\, E_2(\infty) = 0,\, E_3(\infty) = 0$

$$J^{(2)} = 3H\left(\tau + \frac{2}{3}\right)$$

Basic problem of Lambda Iteration: Good in outer layers, but does not work at large optical depths, because exponential integral function approaches zero exponentially.

Exact solution of (*) for Hopf function, e.g., by Laplace transformation (Kourganoff, Basic Methods in Transfer Problems)

Analytical approximation (Unsöld, Sternatmosphären, p. 138)

$$q(\tau) \approx 0.6940 - 0.1167 e^{-1.972\tau}$$

Gray atmosphere: Interpretation of results

Temperature gradient

$$\frac{d}{d\tau}T^4 = 4T^3\frac{dT}{d\tau} = \frac{3}{4}T_{\text{eff}}^4$$

$\frac{dT}{d\tau} \sim T_{\text{eff}}^4$ The higher the effective temperature, the steeper the temperature gradient.

$\frac{dT}{dt} = \kappa\frac{dT}{d\tau}$ The larger the opacity, the steeper the (geometric) temperature gradient.

Flux of gray atmosphere

LTE: $S_\nu = B_\nu(T(\tau))$

$$H_\nu(\tau) = \frac{1}{2}\int_\tau^\infty B_\nu(T(\tau))E_2(t-\tau)dt - \frac{1}{2}\int_0^\tau B_\nu(T(\tau))E_2(\tau-t)dt$$

with $\alpha = h\nu/kT_{\text{eff}}$, $T/T_{\text{eff}} = \left[3/4\,(\tau+q(\tau))\right]^{1/4} = p(\tau) \rightarrow h\nu/kT = \alpha\, p(\tau)$

$H_\alpha d\alpha = H_\nu d\nu$ and $H = \sigma/_{4\pi}T_{\text{eff}}^4$

$$\rightarrow H_\alpha(\tau)/H = \frac{H_\nu}{H}\frac{d\nu}{d\alpha} = \frac{4\pi}{\sigma T_{\text{eff}}^4}\frac{kT_{\text{eff}}}{h}H_\nu = \frac{4\pi k^4}{h^3c^2\sigma}\alpha^3\left(\int_\tau^\infty \frac{E_2(t-\tau)}{\exp(\alpha p(\tau))-1}dt - \int_0^\tau \frac{E_2(\tau-t)}{\exp(\alpha p(\tau))-1}dt\right)$$

$$\frac{1}{2}\frac{2h\nu^3}{c^2}\frac{4\pi}{\sigma}\frac{k}{h}\frac{\alpha^3k^3}{h^3\nu^3}$$

Gray atmosphere: Interpretation of results

Limb darkening of total radiation

$$\mathrm{I}(\tau=0,\mu) = \mathrm{S}(\tau=\mu) = \mathrm{B}(\mathrm{T}(\tau=\mu)) = \frac{\sigma}{\pi}\mathrm{T}^4(\tau=\mu) = \frac{\sigma}{\pi}\mathrm{T}_{\text{eff}}^4\,\frac{3}{4}\left(\mu+\frac{2}{3}\right)$$

$$\rightarrow \frac{\mathrm{I}(0,\mu)}{\mathrm{I}(0,1)} = \frac{\mu+2/3}{1+2/3} = \frac{2}{5}(1+\frac{3}{2}\cos\vartheta)$$

i.e., intensity at limb of stellar disk smaller than at center by 40%, good agreement with solar observations

Empirical determination of temperature stratification

measure $I(\tau=0,\mu) \rightarrow S(\tau=\mu) \rightarrow S(\tau) = B(T(\tau)) \rightarrow T$

Observations at different wavelengths yield different T-structures, hence, the opacity must be a function of wavelength

The Rosseland opacity

Gray approximation (κ=const) very coarse, ist there a good mean value $\bar{\kappa}$? What choice to make for a mean value?

	gray	non-gray
transfer equation	$\mu \frac{dI}{dz} = \kappa(S-I)$	$\mu \frac{dI_\nu}{dz} = \kappa(\nu)(S_\nu - I_\nu)$
0-th moment	$\frac{dH}{dz} = \kappa(S-J) = 0$	$\frac{dH_\nu}{dz} = \kappa(\nu)(S_\nu - J_\nu)$
1st moment	$\frac{dK}{dz} = -\kappa H$	$\frac{dK_\nu}{dz} = -\kappa(\nu) H_\nu$

For each of these 3 equations one can find a mean $\bar{\kappa}$, with which the equations for the gray case are equal to the frequency-integrated non-gray equations.

Because we demand flux conservation, the 1st moment equation is decisive for our choice:

→ Rosseland mean of opacity

The Rosseland opacity

$$\int_0^\infty H_\nu d\nu = \text{const} = \int_0^\infty \frac{1}{\kappa(\nu)} \frac{dK_\nu}{dz} d\nu = \frac{1}{\kappa_R} \frac{dK}{dz}$$

$$\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa(\nu)} \frac{dK_\nu}{dz} d\nu}{\frac{dK}{dz}} \quad \text{with Eddington approximation} \quad K = 1/3J \ \text{ and LTE } \ J = B:$$

$$\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa(\nu)} \frac{dB_\nu}{dz} d\nu}{\frac{dB}{dz}} \quad \text{with} \quad \frac{dB_\nu}{dz} = \frac{dB_\nu}{dT}\frac{dT}{dz} \quad \text{and} \quad \frac{dB}{dz} = \frac{d}{dz}\left(\frac{\sigma}{\pi}T^4\right) = \frac{4\sigma}{\pi}T^3 \frac{dT}{dz}$$

$$\boxed{\frac{1}{\kappa_R} = \frac{\int_0^\infty \frac{1}{\kappa(\nu)} \frac{dB_\nu}{dT} d\nu}{\frac{4\sigma}{\pi}T^3}}$$

Definition of Rosseland mean of opacity

The Rosseland opacity

The Rosseland mean $\frac{1}{\kappa_R}$ is a weighted mean

of opacity $\frac{1}{\kappa(\nu)}$ with weight function $\frac{dB_\nu}{dT}$

Particularly, strong weight is given to those frequencies, where the radiation flux is large.

The corresponding optical depth is called Rosseland depth

$$\tau_{Ross}(z) = \int_0^z \kappa_R(z')dz'$$

For $\tau_{Ross} >> 1$ the gray approximation with κ_R is very good,

i.e. $T^4(\tau_{Ross}) = \frac{3}{4} T_{\text{eff}}^4 (\tau_{Ross} + q(\tau_{Ross}))$

Convection

Compute model atmosphere assuming

- Radiative equilibrium (Sect. VI) $\rightarrow$ temperature stratification
- Hydrostatic equilibrium $\rightarrow$ pressure stratification

Is this structure stable against convection, i.e. small perturbations?

- Thought experiment

Displace a blob of gas by Δr upwards, fast enough that no heat exchange with surrounding occurs (i.e., adiabatic), but slow enough that pressure balance with surrounding is retained (i.e. << sound velocity)

Inside of blob outside

$$T+\Delta T_{\text{ad}}=T_{\text{ad}}(r+\Delta r) \qquad T+\Delta T_{\text{rad}}=T_{\text{rad}}(r+\Delta r)$$
$$\rho+\Delta\rho_{\text{ad}}=\rho_{\text{ad}}(r+\Delta r) \qquad \rho+\Delta\rho_{\text{rad}}=\rho_{\text{rad}}(r+\Delta r)$$

$$\uparrow \Delta r$$

$$T(r),\rho(r) \qquad T(r),\rho(r)$$

$\rho_{\text{ad}}(r+\Delta r)<\rho_{\text{rad}}(r+\Delta r)\rightarrow$ further buoyancy, unstable

$\rho_{\text{ad}}(r+\Delta r)>\rho_{\text{rad}}(r+\Delta r)\rightarrow$ gas blob falls back, stable

i.e. $\left|\frac{d\rho_{\text{ad}}}{dr}\right|\begin{Bmatrix}>\\<\end{Bmatrix}\left|\frac{d\rho_{\text{rad}}}{dr}\right|\begin{cases}\text{unstable}\\\text{stable}\end{cases}$

with ideal gas equation $p=\frac{k}{Am_H}\rho T$ and pressure balance $\rho_{\text{ad}}T_{\text{ad}}=\rho_{\text{rad}}T_{\text{rad}}$

$$\frac{dT_{\text{ad}}}{dr}\begin{Bmatrix}<\\>\end{Bmatrix}\frac{dT_{\text{rad}}}{dr}\begin{cases}\text{unstable}\\\text{stable}\end{cases}$$

Stratification becomes unstable, if temperature gradient dT_{ad}/dr rises above critical value.

Alternative notation

Pressure as independent depth variable:

hydrostatic equation: $dp=-\rho g_{\text{eff}}dr=-\frac{Am_H}{k}g_{\text{eff}}\frac{p}{T}dr$ (ideal gas)

$$\rightarrow dr=-dp\frac{kT}{Am_H g_{\text{eff}}p}$$

$$\frac{dT}{dr}=-\frac{Am_H}{k}g_{\text{eff}}\frac{dT/T}{dp/p}=-\frac{Am_H}{k}g_{\text{eff}}\frac{d(\ln T)}{d(\ln p)}$$

$$\frac{d(\ln T_{\text{ad}})}{d(\ln p)}\begin{Bmatrix}<\\>\end{Bmatrix}\frac{d(\ln T_{\text{rad}})}{d(\ln p)}\begin{cases}\text{unstable}\\\text{stable}\end{cases}$$

Schwarzschild criterion

Abbreviated notation

$$\nabla_{\text{ad}}=\frac{d(\ln T_{\text{ad}})}{d(\ln p)};\nabla_{\text{rad}}=\frac{d(\ln T_{\text{rad}})}{d(\ln p)}$$

$\nabla_{\text{ad}}>\nabla_{\text{rad}}$ stable

The adiabatic gradient

$dQ = 0$ (no heat exchange)

$dQ = dE + pdV$ (1st law of thermodynamics)

$dE = c_V dT$ internal energy $\Rightarrow c_V dT + pdV = 0$ (*)

Internal energy of a one-atomic gas excluding effects of ionisation and excitation

$$E = \frac{3}{2} NkT \rightarrow c_V = \frac{3}{2} Nk$$

But if energy can be absorbed by ionization:

$$c_V >> \frac{3}{2} Nk$$

Specific heat at constant pressure

$$c_p = \left.\frac{\partial Q}{\partial T}\right|_{p=const} = \left.\frac{dE}{dT} + p\frac{dV}{dT}\right|_{p=const} = c_V + p\frac{d(NkT/p)}{dT} = c_V + p\frac{Nk}{p}$$

$$\rightarrow c_p - c_V = Nk$$

The adiabatic gradient

Ideal gas: $pV = NkT \Rightarrow Vdp + pdV = NkdT = \left(c_p - c_V\right) dT$

$$dT = \frac{Vdp + pdV}{c_p - c_V} \quad (**)$$

$$\text{from(*) with (**)} \quad \rightarrow c_V \frac{Vdp + pdV}{c_p - c_V} + pdV = 0 \quad \Bigg| \begin{array}{l} /pV \\ \cdot \dfrac{c_p - c_V}{c_V} \end{array}$$

$$\frac{dp}{p} + \frac{dV}{V} + \frac{dV}{V}\frac{c_p - c_V}{c_V} = 0$$

$$\frac{dp}{p} + \frac{dV}{V}\frac{c_p}{c_V} = 0$$

$$\frac{c_p}{c_V} d(\ln V) = -d(\ln p)$$

$$\text{definition:} \quad \gamma := \frac{c_p}{c_V} \quad \frac{d(\ln V)}{d(\ln p)} = -\frac{1}{\gamma}$$

The adiabatic gradient

needed: $\left.\dfrac{d(\ln T)}{d(\ln p)}\right|_{\mathrm{ad}}$

$$T = pV/Nk$$

$$\ln T = \ln p + \ln V - \ln(Nk)$$

$$\frac{d(\ln T)}{d(\ln p)} = 1 + \frac{d(\ln V)}{d(\ln p)}$$

$$\frac{d(\ln T)}{d(\ln p)} = 1 - \frac{1}{\gamma} = \frac{\gamma-1}{\gamma}$$

$$\nabla_{\mathrm{ad}} = \frac{\gamma-1}{\gamma}$$

$\nabla_{\mathrm{rad}} < \dfrac{\gamma-1}{\gamma}$ stable Schwarzschild criterion

The adiabatic gradient

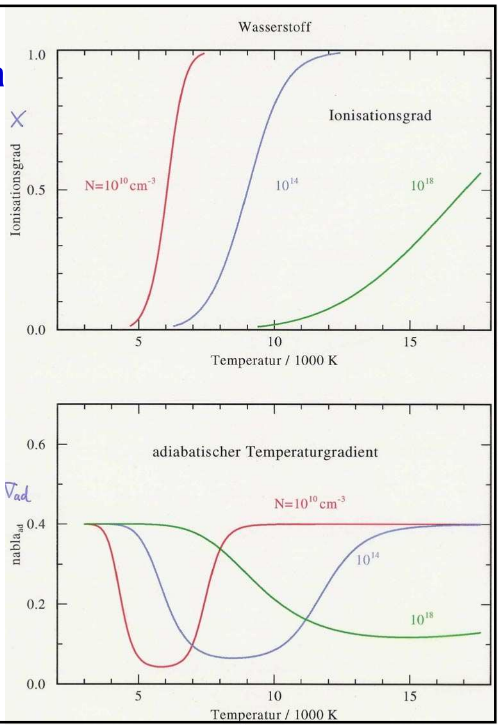
- 1-atomic gas $c_{\mathrm{V}} = 3/2\,Nk \quad c_{\mathrm{p}} = c_{\mathrm{V}} + Nk = 5/2\,Nk$
  $\gamma = 5/3 \quad \nabla_{\mathrm{ad}} = 2/5 = 0.4$
- with ionization $\gamma \to 1 \quad \nabla_{\mathrm{ad}} \to 0$ convection starts $\gamma-$effect
- Most important example: Hydrogen (Unsöld p.228)

$$\nabla_{\mathrm{ad}} = \frac{2 + \left(x - x^2\right)\left(5/2 + E_{\mathrm{Ion}}/kT\right)}{5 + \left(x - x^2\right)\left(5/2 + E_{\mathrm{Ion}}/kT\right)^2}$$

with ionization degree $x = -\dfrac{f(T)}{2N} + \sqrt{\left(\dfrac{f(T)}{2N}\right)^2 + \dfrac{f(T)}{N}}$

Stellar Atmospheres: Radiative Equilibrium

The adiaba



$$\nabla_{\text{ad}} = \frac{2+\left(x-x^2\right)\left(5/2+E_{\text{Ion}}/kT\right)}{5+\left(x-x^2\right)\left(5/2+E_{\text{Ion}}/kT\right)^2}$$

$$x = -\frac{f(T)}{2N} + \sqrt{\left(\frac{f(T)}{2N}\right)^2 + \frac{f(T)}{N}}$$

Stellar Atmospheres: Radiative Equilibrium

Example: Grey approximation

$$T^4(\tau) = \tfrac{3}{4} T_{\text{eff}}^4 \left(\tau + \tfrac{2}{3}\right)$$

$$4 \ln T = \ln\left(\tfrac{3}{4} T_{\text{eff}}^4\right) + \ln\left(\tau + \tfrac{2}{3}\right)$$

$$\frac{d(\ln T)}{d\tau} = \frac{d\left(\ln\left(\tau + \tfrac{2}{3}\right)\right)}{4 d\tau} = \frac{1}{4\left(\tau + \tfrac{2}{3}\right)}$$

hydrostatic equation: $\frac{dp}{d\tau} = \frac{g}{\kappa}$ Ansatz: $\kappa = Ap^b$ (κ here a mass absorption coefficient)

$$\rightarrow p^b \frac{dp}{d\tau} = \frac{g}{A} \quad \text{integrate} \rightarrow \frac{1}{b+1} p^{b+1} = \frac{g}{A}\tau \qquad \rightarrow \frac{g}{Ap^{b+1}} = \frac{1}{(b+1)\tau}$$

$$\frac{d(\ln p)}{d\tau} = \frac{1}{p}\frac{dp}{d\tau} = \frac{1}{p}\frac{g}{Ap^b} = \frac{g}{Ap^{b+1}} = \frac{1}{(b+1)\tau}$$

$$\nabla_{\text{rad}} = \frac{d\ln T/d\tau}{d\ln p/d\tau} = \frac{(b+1)\tau}{4\left(\tau + \tfrac{2}{3}\right)}$$

∇_{rad} becomes large, if opacity strongly increases with depth (i.e. exponent b large).

The absolute value of κ is not essential but the change of κ with depth (gradient)

∇_{rad} large ($>\nabla_{\text{ad}}$): convection starts, κ-Effekt

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Hydrogen convection zone in the Sun

κ-effect and γ-effect act together

Going from the surface into the interior: At T~6000K ionization of hydrogen begins

∇_{ad} decreases and κ increases, because a) more and more electrons are available to form H^- and b) the excitation of H is responsible for increased bound-free opacity

In the Sun: outer layers of atmosphere radiative

inner layers of atmosphere convective Video

In F stars: large parts of atmosphere convective

In O,B stars: Hydrogen completely ionized, atmosphere radiative; He I and He II ionization zones, but energy transport by convection inefficient

Transport of energy by convection

Consistent hydrodynamical simulations very costly;

Ad hoc theory: mixing length theory (Vitense 1953)

Model: gas blobs rise and fall along distance *l* (mixing length). After moving by distance *l* they dissolve and the surrounding gas absorbs their energy.

$$l = \alpha H(r) \quad H = \text{pressure scale height}$$

$$\alpha \ \text{ mixing length parameter}$$

$$\alpha = 0.5 \cdots 2$$

Gas blobs move without friction, only accelerated by buoyancy; detailed presentation in Mihalas‘ textbook (p. 187-190)

Transport of energy by convection

Again, for details see Mihalas (p. 187-190)

For a given temperature structure

$\rightarrow$ compute $F_{\text{conv}}(r)$

$\rightarrow$ flux conservation including convective flux

$$F_{\text{rad}}(r) = \frac{\sigma}{\pi} T_{\text{eff}}^4 - F_{\text{conv}}(r)$$

$\rightarrow$ new temperature stratification $T(r)$

with $\nabla_{\text{ad}} < \nabla < \nabla_{\text{rad}}$

iterate

Summary: Radiative Equilibrium

Radiative Equilibrium:

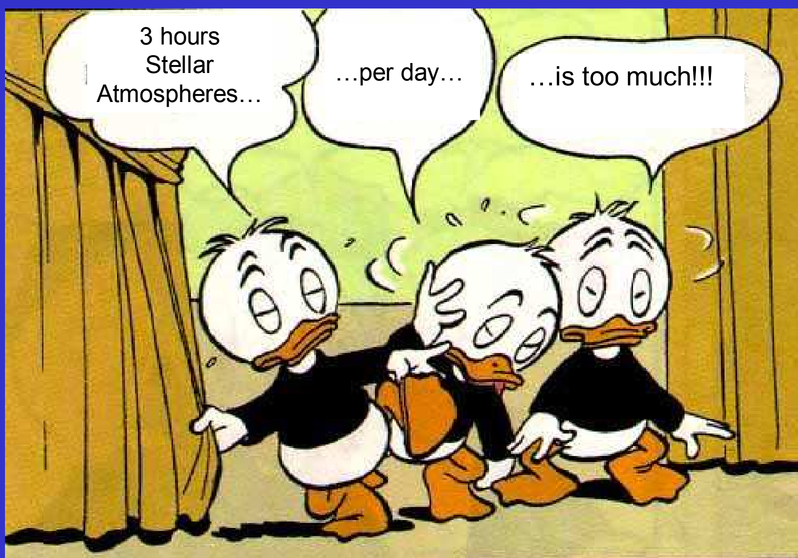
$$A \cdot \left[\int_0^\infty \kappa (J_\nu - S_\nu) d\nu \right] + B \cdot \left[\int_0^\infty \frac{d(f_\nu J_\nu)}{d\tau} d\nu - H \right] = 0$$

Schwarzschildt Criterion:

$$\frac{d(\ln T_{\mathrm{ad}})}{d(\ln p)} \begin{Bmatrix} < \\ > \end{Bmatrix} \frac{d(\ln T_{\mathrm{rad}})}{d(\ln p)} \begin{cases} \text{unstable} \\ \text{stable} \end{cases}$$

Temperature of a gray Atmosphere

$$T^4 = \frac{3}{4} T_{\mathrm{eff}}^4 \left(\tau + \frac{2}{3} \right)$$



The non-LTE Rate Equations

Statistical equations

Population numbers

LTE: population numbers follow from Saha-Boltzmann equations, i.e. purely local problem

$$n_i^* = n_i^* (T, n_e)$$

Non-LTE: population numbers also depend on radiation field. This, in turn, is depending on the population numbers in all depths, i.e. non-local problem.

$$n_i = n_i (T, n_e, J)$$

The Saha-Boltzmann equations are replaced by a detailed consideration of atomic processes which are responsible for the population and de-population of atomic energy levels:

$$\left.\begin{array}{l}\text{Excitation and de-excitation}\\ \\ \text{Ionization and recombination}\end{array}\right\}\text{ by radiation or collisions}$$

Statistical Equilibrium

Change of population number of a level with time:
= Sum of all population processes into this level
- Sum of all de-population processes out from this level

$$\frac{d}{dt}n_i = \sum_{j \neq i} n_j P_{ji} - n_i \sum_{j \neq i} P_{ij}$$

$\frac{d}{dt}n_i$ $= \sum_{j \neq i} n_j P_{ji}$ $- n_i \sum_{j \neq i} P_{ij}$

One such equation for each level

The transition rate P_{ij} comprises radiative rates R_{ij}

and collision rates C_{ij}

In stellar atmospheres we often have the stationary case:

$$\frac{d}{dt}n_i = 0 \quad \text{hence} \quad \boxed{\sum_{j \neq i} n_j P_{ji} = n_i \sum_{j \neq i} P_{ij}} \quad \text{for all levels } i$$

These equations determine the population numbers.

Stellar Atmospheres: Non-LTE Rate Equations

Radiative rates: bound-bound transitions

Two alternative formulations:

a) Einstein coefficients $B_{ij}\ B_{ji}\ A_{ji}$

b) Line absorption coefficients $\sigma_{ij}(\nu)$

advantage a): useful for analytical expressions with simplified model atoms

advantage b): similar expressions in case of bound-free transitions: good for efficient programming

Number of transitions i→j induced by intensity I_ν in frequency interval $d\nu$ und solid angle $d\omega$

$$n_i B_{ij} \varphi_\nu I_\nu d\nu d\omega / 4\pi \quad \text{(absorbed Energy / } h\nu\text{)}$$

Integration over frequencies and angles yields

$$n_i R_{ij} = n_i B_{ij} \int_0^\infty \varphi_\nu J_\nu d\nu$$

Or alternatively

$$\text{with } \sigma_{ij}(\nu) = B_{ij}\varphi(\nu) h\nu / 4\pi \quad \boxed{n_i R_{ij} = n_i 4\pi \int_0^\infty \frac{\sigma_{ij}(\nu)}{h\nu} J_\nu d\nu}$$

Radiative rates: bound-bound transitions

In analogy, number of stimulated emissions:

$$n_j R'_{ji} = n_j B_{ji} \int_0^\infty \varphi_\nu J_\nu d\nu = n_j B_{ij} \frac{g_i}{g_j} \int_0^\infty \varphi_\nu J_\nu d\nu$$

$$\boxed{n_j R'_{ji} = n_j 4\pi \frac{g_i}{g_j} \int_0^\infty \frac{\sigma_{ij}}{h\nu} J_\nu d\nu}$$

Number of spontaneous emissions:

$$n_j R''_{ji} = n_j \int_0^\infty A_{ji} \varphi_\nu d\nu = n_j B_{ji} \int_0^\infty \frac{2h\nu^3}{c^2} \varphi_\nu d\nu$$

$$\boxed{n_j R''_{ji} = n_j 4\pi \frac{g_i}{g_j} \int_0^\infty \frac{\sigma_{ij}}{h\nu} \frac{2h\nu^3}{c^2} d\nu}$$

Total downwards rate:

$$n_j \bar{R}_{ji} = n_j \left(R'_{ji} + R''_{ji} \right) = n_j 4\pi \frac{g_i}{g_j} \int_0^\infty \frac{\sigma_{ij}}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu \right) d\nu$$

$$\boxed{n_j \bar{R}_{ji} = n_j \left(\frac{n_i}{n_j} \right)^* R_{ji} = n_j \left(\frac{n_i}{n_j} \right)^* [4\pi] \int_0^\infty \frac{\sigma_{ij}}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu \right) e^{-h\nu/kT} d\nu}$$

Radiative rates: bound-free transitions

Also possible: ionization into excited states of parent ion

Example C III:

Ground state $2s^2\ ^1S$

Photoionisation produces C IV in ground state $2s\ ^2S$

C III in first excited state $2s2p\ ^3P^o$

Two possibilities:

Ionization of 2p electron → C IV in ground state $2s\ ^2S$

Ionization of 2s electron → C IV in first excited state $2p\ ^2P$

C III two excited electrons, e.g. $2p^2\ ^3P$

Photoionization only into excited C IV ion $2p\ ^2P$

Radiative rates: bound-free transitions

Number of photoionizations = absorbed energy in dν, divided by photon energy, integrated over frequencies and solid angle

$$\int_0^\infty \oint n_i p_\nu I_\nu \, d\omega d\nu \rightarrow \boxed{n_i R_{ij} = n_i 4\pi \int_0^\infty \frac{\sigma_{ij}(\nu)}{h\nu} J_\nu d\nu}$$

Number of spontaneous recombinations:

$$\int_0^\infty \oint n_j n_e(\mathrm{v}) F(\mathrm{v}) \, d\omega \mathrm{v} d\nu \rightarrow n_j R_{ji} = n_j 4\pi \int_0^\infty n_e(\mathrm{v}) \frac{2h\nu^3}{c^2} G(\mathrm{v}) \frac{h}{m} d\nu$$

$$n_j R_{ji} = n_j 4\pi \int_0^\infty n_e(\mathrm{v}) \frac{2h\nu^3}{c^2} p_\nu \frac{m}{h} e^{-h\nu/kT} \left(\frac{n_i}{n_j}\right)^* \frac{1}{n_e(\mathrm{v})} \frac{h}{m} d\nu$$

$$\boxed{n_j R_{ji} = n_j \left(\frac{n_i}{n_j}\right)^* 4\pi \int_0^\infty \frac{\sigma_{ij}(\nu)}{h\nu} \frac{2h\nu^3}{c^2} e^{-h\nu/kT} d\nu}$$

Radiative rates: bound-free transitions

Number of induced recombinations

$$\int_0^\infty \oint n_j n_e(\mathrm{v}) G(\mathrm{v}) I_\nu \, d\omega \mathrm{v} d\nu \rightarrow n_j R_{ji} = n_j 4\pi \int_0^\infty n_e(\mathrm{v}) G(\mathrm{v}) J_\nu \frac{h}{m} d\nu$$

$$n_j R_{ji} = n_j 4\pi \int_0^\infty n_e(\mathrm{v}) p_\nu \frac{m}{h} e^{-h\nu/kT} \left(\frac{n_i}{n_j}\right)^* \frac{1}{n_e(\mathrm{v})} J_\nu \frac{h}{m} d\nu$$

$$\boxed{n_j R_{ji} = n_j \left(\frac{n_i}{n_j}\right)^* 4\pi \int_0^\infty \frac{\sigma_{ij}(\nu)}{h\nu} J_\nu e^{-h\nu/kT} d\nu}$$

Total recombination rate

$$\boxed{n_j \left(\frac{n_i}{n_j}\right)^* R_{ji} = n_j \left(\frac{n_i}{n_j}\right)^* 4\pi \int_0^\infty \frac{\sigma_{ij}(\nu)}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu\right) e^{-h\nu/kT} d\nu}$$

Radiative rates

Upward rates:

$n_i R_{ij}$ with

$$R_{ij} = 4\pi \int_0^\infty \frac{\sigma_{ij}(\nu)}{h\nu} J_\nu d\nu$$

Downward rates:

$n_j \left(\frac{n_i}{n_j}\right)^* R_{ji}$ with

$$R_{ji} = 4\pi \int_0^\infty \frac{\sigma_{ij}(\nu)}{h\nu} \left(\frac{2h\nu^3}{c^2} + J_\nu\right) e^{-h\nu/kT} d\nu$$

Remark: in TE we have $J_\nu = B_\nu \Rightarrow R_{ij}^* = R_{ji}^* \Rightarrow \frac{n_i}{n_j} = \left(\frac{n_i}{n_j}\right)^*$

Collisional rates

Stellar atmosphere: Plasma, with atoms, ions, electrons

Particle collisions induce excitation and ionization

Cool stars: matter mostly neutral $\Rightarrow$ frequent collisions with neutral hydrogen atoms

Hot stars: matter mostly ionized $\Rightarrow$ collisions with ions become important; but much more important become electron collisions

$$\frac{\mathrm{v}_{electron}}{\mathrm{v}_{ion}} = \left(\frac{\text{ion mass}}{\text{electron mass}}\right)^{1/2} = \left(\frac{m_H A}{m_e}\right)^{1/2} \approx 43\sqrt{A}$$

Therefore, in the following, we only consider collisions of atoms and ions with electrons.

Electron collisional rates

Transition $i \rightarrow j$ (j: bound or free), $\sigma_{ij}(\mathrm{v})$ = electron collision cross-section, v = electron speed

Total number of transitions $i \rightarrow j$:

$$n_i C_{ij} = n_i n_e \int_{\mathrm{v}_0}^{\infty} \sigma_{ij}(\mathrm{v}) f(\mathrm{v}) \mathrm{v} d\mathrm{v} = n_i n_e \Omega_{ij}(T)$$

v_0 minimum velocity necessary for excitation (threshold)

$f(\mathrm{v})d\mathrm{v}$ velocity distribution (Maxwell)

In TE we have therefore

$$\mathrm{n}_i^* C_{ij} = \mathrm{n}_j^* C_{ji}$$

Total number of transitions $j \rightarrow i$:

$$\mathrm{n}_j C_{ji} = \mathrm{n}_j \left(\frac{n_i}{n_j} \right)^* C_{ij}$$

Electron collisional rates

We look for: collisional cross-sections $\sigma_{ij}(\mathrm{v})$

- experiments
- quantum mechanical calculations

Usually: Bohr radius πa_0^2 as unit for cross-section $\sigma_{ij}(\mathrm{v})$

$\sigma_{ij}(\mathrm{v}) = \pi a_0^2 \, Q_{ij}$

Q_{ij} usually tabulated as function of energy of colliding electron

$$\Omega_{ij}(T) = \int_{\mathrm{v}_0}^{\infty} \sigma_{ij}(\mathrm{v}) f(\mathrm{v}) \mathrm{v} d\mathrm{v} \quad \text{with} \quad 1/2\, m\mathrm{v}^2 = E \quad \text{and} \quad f(\mathrm{v})d\mathrm{v} = \left(\frac{\mathrm{m_e}}{2\pi \mathrm{kT}} \right)^{1/2} e^{-m\mathrm{v}^2/2kT} 4\pi \mathrm{v}^2 d\mathrm{v}$$

$$= C_0 \sqrt{T} \int_{\mathrm{u}_0}^{\infty} Q_{ij}(\mathrm{ukT}) u e^{-u} d\mathrm{u} \quad \text{with} \quad \mathrm{u} := \mathrm{E}/kT \quad \text{and} \quad C_0 = \pi a_0^2 \left(\frac{8k}{m_e \pi} \right)^{1/2} = 5.456 \cdot 10^{-11}$$

$$\boxed{\Omega_{ij}(T) = C_0 \sqrt{T} e^{-u_0} \Gamma_{ij}(T)} \quad \text{with} \quad \Gamma_{ij}(T) = \int_0^{\infty} Q_{ij}(E_0 + \mathrm{xkT})(x + u_0) e^{-x} d\mathrm{x}, \quad \mathrm{x} := (\mathrm{E}/kT - \mathrm{E}_0/kT)$$

Electron collisional rates

$$\boxed{\Omega_{ij}(T) = C_0 \sqrt{T} e^{-u_0} \Gamma_{ij}(T)}$$

$$n_i C_{ij} = n_i n_e \Omega_{ij}(T)$$

Advantage of this choice of notation:
Main temperature dependence is described by $\sqrt{T} e^{-E_0/kT}$
$\Gamma_{ij}(T)$ only weakly varying function of T
Hence, simple polynomial fit possible
$\Rightarrow$ Important for numerical application

Now: examples how to compute the C_{ij}

Computation of collisional rates: Excitation

Van Regemorter (1962): Very useful approximate formula for allowed dipole transitions

$$\boxed{C_{ij} = C_0 n_e \sqrt{T} 14.5 f_{ij} \left(\frac{E_H}{E_0}\right)^2 u_0 e^{-u_0} \Gamma(u_0)}$$

E_H hydrogen ionization energy

f_{ij} oscillator strength of radiative transition

$E_0 = h\nu_{ij} \quad u_0 = E_0 / kT \quad \Gamma(u_0) = \max\left[\bar{g}, 0.276 e^{u_0} E_1(u_0)\right]$

$$\bar{g} = \begin{cases} 0.7 & \text{for transitions between levels with equal principal quantum number} \\ 0.2 & \text{else} \end{cases}$$

There exist many formulae, made for particular ions and transitions, e.g., (optically) forbidden transitions between n=2 levels in He I (Mihalas & Stone 1968)

$$\boxed{C_{ij} = C_0 n_e \sqrt{T} e^{-u_0} \Gamma(T)}$$

with $\log \Gamma = c_0 + c_1 \log T + c_{-2} (\log T)^{-2}$

coefficients **c** tabulated for each transition

Computation of collisional rates: Ionization

The Seaton formula is in analogy to the van-Regemorter formula in case of excitation. Here, the photon absorption cross-section for ionization is utilized:

$$\boxed{C_{ij} = 1.55 \cdot 10^{13} \sigma_0 \bar{g} \frac{n_e}{\sqrt{T}} \frac{e^{-u_0}}{u_0}}$$

$\sigma_0 =$ threshold photon cross-section for ionization

$$\bar{g} = \begin{cases} 0.1 & \text{for ions with charge } Z = 1 \\ 0.2 & \text{for ions with charge } Z = 2 \\ 0.3 & \text{for ions with charge } Z > 2 \end{cases}$$

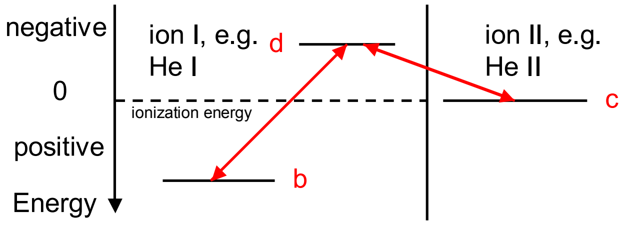
Alternative: semi-empirical formula by Lotz (1968):

$$\boxed{C_{ij} = C_0 n_e \sqrt{T}\, 2.5a \left(\frac{E_H}{E_0}\right)^2 u_0 \left[E_1(u_0) - b e^c u_0 E_1(u_1)/u_1\right]}$$

$u_1 = u_0 + c \quad a,b,c$ empirical quantities, adjusted to individual atoms

For H und He specific fit formulae are used, mostly from Mihalas (1967) and Mihalas & Stone (1968)

Autoionization and dielectronic recombination



b bound state, d doubly excited state, autoionization level

c ground state of next Ion

d $\rightarrow$ c: Autoionization. d decays into ground state of next ionization stage plus free electron

c $\rightarrow$ d $\rightarrow$ b: Dielectronic recombination. Recombination via a doubly excited state of next lower ionization stage. d auto-ionizes again with high probability: $A_{auto}=10^{13}...10^{14}$/sec! But sometimes a stabilizing transition d $\rightarrow$ b occurs, by which the excited level decays radiatively.

Computation of rates

Number of dielectronic recombinations from c to b:

$$n_c R_{cb} = n_d A_s \quad A_s = \text{probability for spontaneous stabilizing transition}$$

In the limit of weak radiation fields the reverse process can be neglected. Then we obtain (Bates 1962):

$$n_d = n_d^* A_a / (A_a + A_s) \quad \text{with } n_d^* = n_c n_e C_1 T^{-3/2} e^{E^{\mathrm{d}}_{\mathrm{Ion}}/kT} = n_c n_e \Phi_{cd}(T)$$

$$A_a = \text{transition probability for autoionization}$$

So, the number of dielectronic recombinations from c to b is:

$$\boxed{n_c R_{cb} = n_c n_e \Phi_{cd}(T) A_s A_a / (A_a + A_s)}$$

Computation of rates

There are two different regimes:

a) high temperature dielectronic recombination **HTDR**

b) low temperature dielectronic recombination **LTDR**

for the cases that the autoionizing levels are close to the ionization limit (b) or far above it (a)

a) Most important recombination process He II → He I in the solar corona (T~2·10^6K)

b) Very important for specific ions in photospheres (T< 10^5K) e.g. N III λ4634-40Å emission complex in Of stars

Reason: upper level is overpopulated, because a stabilizing transition is going into it.

Because in case b) $A_a >> A_s \rightarrow n_d = n_d^*$

LTDR

The radiation field in photospheres is **not** weak, i.e., the reverse process b → d is induced

Recombination rate:

$$\boxed{n_c R_{cb} = n_c n_e \Phi_{cd}(T) A_s \left(1 + \frac{c^2}{2h\nu^3} \bar{J}\right)}$$

$\bar{J}$ mean intensity in stabilizing transition, i.e.,
given by continuum value (line very broad, because short lifetime)

Reverse process:

$$\boxed{n_b R_{bc} = n_b B_{bd} \bar{J} = n_b A_s \frac{c^2}{2h\nu^3} \frac{g_d}{g_b} \bar{J}}$$

These rates are formally added to the usual ionization and recombination rates and do not show up explicitly in the rate equations.

Complete rate equations

For each atomic level ***i*** of each ion, of each chemical element we have:

$$n_i \sum_{j \neq i} P_{ij} - \sum_{j \neq i} n_j P_{ji} = 0$$

In detail:

$$n_i \left[\sum_{j>i} \left(R_{ij} + C_{ij} \right) + \sum_{j<i} \left(\frac{n_j}{n_i} \right)^* \left(R_{ij} + C_{ji} \right) \right] - \sum_{j>i} n_j \left(\frac{n_i}{n_j} \right)^* \left(R_{ji} + C_{ij} \right) - \sum_{j<i} n_j \left(R_{ji} + C_{ji} \right) = 0$$

excitation and ionization

rates out of i

de-excitation and recombination

de-excitation and recombination

rates into i

excitation and ionization

Closure equation

One equation for each chemical element is redundant, e.g., the equation for the highest level of the highest ionization stage; to see this, add up all equations except for the final one: these rate equations only yield population **ratios**.

We therefore need a closure equation for each chemical species:

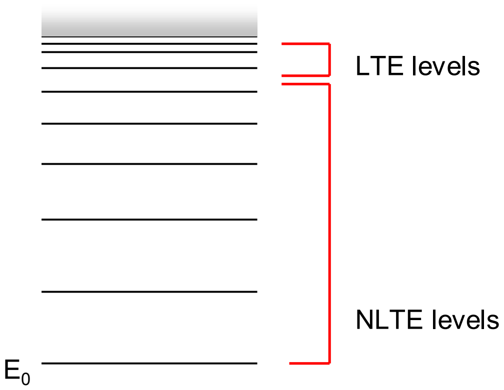
Abundance definition equation of species k, written for example as number abundance y_k relative to hydrogen:

$$y_k = \frac{\sum \text{population numbers of species k}}{\sum \text{population numbers of hydrogen}}$$

Abundance definition equation

Notation:

Population number of level i in ionization stage l: $n_{l,i}$



do not appear explicitly in the rate equations; populations depend on ground level of next ionization stage:

$$n_{l,i}^{*} = n_e n_{l+1,1} \Phi_{l,i}(T)$$

Abundance definition equation

Notation:

NION number of ionization stages of chemical element *k*

NL(l) number of NLTE levels of ion *l*

LTE(l) number of LTE levels of ion *l*

$$\sum_{l=1}^{NION}\left[\sum_{i=1}^{NL(l)} n_{l,i} + \sum_{i=1}^{LTE(l)} n^*_{l,i}\right] = y_k\left[\sum_{i=1}^{NL(H)} n_i + \sum_{i=1}^{LTE(H)} n^*_i + n_{protons}\right] \Rightarrow$$

$$\boxed{\sum_{l=1}^{NION}\left[\sum_{i=1}^{NL(l)} n_{l,i} + n_{l+1,1} n_e \sum_{i=1}^{LTE(l)} \Phi_{li}(T)\right] = y_k\left[\sum_{i=1}^{NL(H)} n_{l,i} + n_{protons}\left(1 + n_e \sum_{i=1}^{LTE(H)} \Phi_i(T)\right)\right]}$$

Also, one of the abundance definition equations is redundant, since abundances are given relative to hydrogen (other definitions don‘t help) ⇒ charge conservation

Charge conservation equation

Notation:

Population number of level ***i***, ion ***l***, element ***k***: n_{kli}

NELEM number of chemical elements

q(l) charge of ion ***l***

$$\sum_{k=1}^{NELEM}\sum_{l=1}^{NION} q(l)\left[\sum_{i=1}^{NL(l,k)} n_{kli} + \sum_{i=1}^{LTE(l,k)} n^*_{kli}\right] = n_e \Rightarrow$$

$$\boxed{\sum_{k=1}^{NELEM}\sum_{l=1}^{NION} q(l)\left[\sum_{i=1}^{NL(l,k)} n_{kli} + n_{k,l+1,1} n_e \sum_{i=1}^{LTE(l)} \Phi_{kli}(T)\right] = n_e}$$

Complete rate equations: Matrix notation

Vector of population numbers

$\underline{n} = (n_1, n_2, \cdots, n_{\text{NLALL}})$ NLALL= total number of NLTE levels

$\underline{\underline{A}}\,\underline{n} = \underline{b}$ rate equation in matrix notation

One such system of equations per depth point

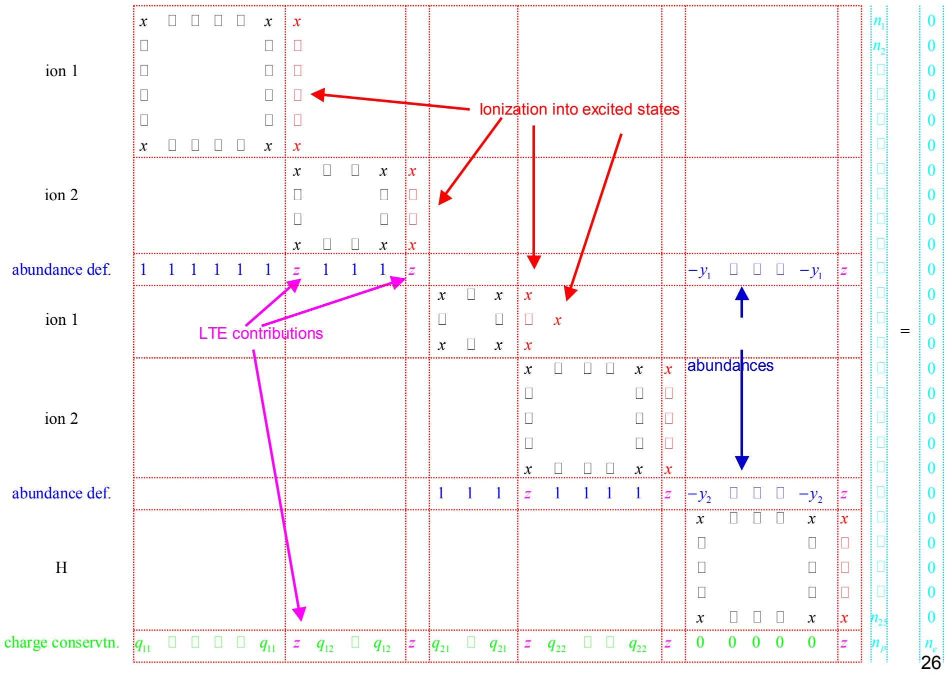
Example: 3 chemical elements

Element 1: NLTE-levels: ion1: 6, ion2: 4, ion3: 1

Element 2: NLTE-levels: ion1: 3, ion2: 5, ion3: 1

Element 3: NLTE-levels: ion1: 5, ion2: 1, hydrogen

Number of levels: NLALL=26, i.e. 26 x 26 matrix



Elements of rate matrix

For each ion *l* with *NL(l)* NLTE levels one obtains a sub-matrix with the following elements:

$$A_{ij} = \begin{cases} -\left(R_{ji} + C_{ji}\right) & j < i \ \text{ lower left} \\ -\left(\frac{n_i}{n_j}\right)^{*}\left(R_{ji} + C_{ij}\right) & j > i \ \text{ upper right} \\ \sum\limits_{m<i}\left(\frac{n_m}{n_i}\right)^{*}\left(R_{im} + C_{mi}\right) + \sum\limits_{m>1}^{k}\left(R_{im} + C_{im}\right) & j = i \ \text{ diagonal} \end{cases}$$

$i = 1 \ldots NL(l) \quad j = 1 \ldots NL(l) \ldots k \quad k$ highest level in parent ion, into which ion l can ionize; does not have to be $= NL(l)+1$!

Elements corresponding to abundance definition eq.

Are located in final row of the respective element:

$$i = \sum_{l=1}^{NION} NL(l)$$

$$A_{ij} = \begin{cases} 1 & j = 1 \ldots \sum\limits_{l=1}^{NION} NL(l) \ \text{except of ground state of excited ions} \\ 1 + n_e \sum\limits_{m=LTE(l-1)} \Phi_{l-1,m} & j = \ \text{ground state of excited ions} \\ -y_k & j = [NLALL\text{ - }NL(H)] \ldots [NLALL - 1] \\ -y_k\left(1 + n_e \sum\limits_{m=LTE(H)} \Phi_m\right) & j = NLALL \end{cases}$$

Elements corresponding to charge conservation eq.

Are located in the very final row of rate matrix, i.e., in

$$i = NLALL$$

$$A_{ij} = \begin{cases} q(l) & j = 1 \ldots NLALL, \quad \text{except of} \\ & \text{ground state of excited ions} \\ q(l) + q(l-1) \sum\limits_{m=LTE(-1)} \Phi_{l-1,m} & \text{else} \end{cases}$$

Note: the inhomogeneity vector **b** (right-hand side of statistical equations) contains zeros except for the very last element (i=NLALL): electron density $\mathbf{n_e}$ (from charge conservation equation)

Solution by linearization

The equation system $\underline{\underline{A}}\,\underline{n} = \underline{b}$ is a linear system for $\underline{n}$ and can be solved if, n_e, T, J_ν are known. But: these quantities are in general unknown. Usually, only approximate solutions within an iterative process are known.

Let all these variables change by $\delta n_e, \delta T, \delta J_\nu$ e.g. in order to fulfill energy conservation or hydrostatic equilibrium.

Response of populations $\delta \underline{n}$ on such changes:

Let $\underline{\underline{A}}\,\underline{n} = \underline{b}$ with actual quantities

And $\left(\underline{\underline{A}} + \delta \underline{\underline{A}}\right)\left(\underline{n} + \delta \underline{n}\right) = \left(\underline{b} + \delta \underline{b}\right)$ with new quantities n_e, T, J_ν

Neglecting 2nd order terms, we have:

$$\underline{\underline{A}}\,\underline{n} - \underline{b} = -\delta \underline{n} - \underline{n}\,\delta \underline{\underline{A}} + \delta \underline{b}$$

Linearization of rate equations

Needed: expressions for: $\delta \underline{\underline{A}}, \delta \underline{b}$

$J\nu$ discretized in NF frequency points

One possibility: $$\boxed{\delta \underline{\underline{A}} = \frac{\partial \underline{\underline{A}}}{\partial T}\delta T + \frac{\partial \underline{\underline{A}}}{\partial n_e}\delta n_e + \sum_{k=1}^{NF}\frac{\partial \underline{\underline{A}}}{\partial J_k}\delta J_k}$$

If in addition to $\underline{n}$ the variables n_e, T, J_k are introduced as unknowns, then we have the

$\rightarrow$ Method of Complete Linearization

Other possibility: eliminates J_k from the equation system by expressing J_k through the other variables n_e, T :

$J_k = f\left(\underline{n}, T, n_e\right)$

As an approximation one uses $J_k^d \sim S_k^d(\underline{n}, T, n_e)$

(and iterates for exact solution)

Linearization of rate equations

$$\boxed{\begin{aligned}\delta \underline{\underline{A}} &= \frac{\partial \underline{\underline{A}}}{\partial T}\delta T + \frac{\partial \underline{\underline{A}}}{\partial n_e}\delta n_e + \sum_{k=1}^{NF}\frac{\partial \underline{\underline{A}}}{\partial S_k}\delta S_k \\ \delta S_k &= \frac{\partial S_k}{\partial T}\delta T + \frac{\partial S_k}{\partial n_e}\delta n_e + \sum_{j=1}^{NLALL}\frac{\partial S_k}{\partial n_j}\delta n_j\end{aligned}}$$

Method of approximate Λ-operators (Accelerated Lambda Iteration)

analogous, $\delta \underline{b}$:

$$\delta \underline{b} = \frac{\partial \underline{b}}{\partial T}\delta T + \frac{\partial \underline{b}}{\partial n_e}\delta n_e + \sum_{k=1}^{NF}\frac{\partial \underline{b}}{\partial S_k}\delta S_k = \left(0, \cdots, 0, \delta n_e\right)$$

Linearization of rate equations

$$\underline{\underline{A}}\,\underline{n} - \underline{b} = -\delta \underline{n}\,\underline{\underline{A}} \quad + \delta T\left[-\frac{\partial \underline{\underline{A}}}{\partial T}\underline{n} - \sum_{k=1}^{NF} \underline{n}\frac{\partial \underline{\underline{A}}}{\partial S_k}\frac{\partial S_k}{\partial T} \right]$$
$$+\delta n_e\left[-\frac{\partial \underline{\underline{A}}}{\partial n_e}\underline{n} - \sum_{k=1}^{NF} \underline{n}\frac{\partial \underline{\underline{A}}}{\partial S_k}\frac{\partial S_k}{\partial n_e} \right]$$
$$+\sum_{j=1}^{NLALL} \delta n_j \left[-\sum_{k=1}^{NF} \underline{n}\frac{\partial \underline{\underline{A}}}{\partial S_k}\frac{\partial S_k}{\partial n_j} \right]$$

Linearized equation for response $\delta \underline{n}$ as answer on changes $\delta n_e, \delta T, \delta J_\nu$

Expressions $\sum_{k=1}^{NF} \underline{n}\frac{\partial \underline{\underline{A}}}{\partial S_k}\frac{\partial S_k}{\partial n_j}$ show the complex coupling of all variables. A change in the radiation field and, hence, the source function at any frequency causes a change of populations of all levels, even if a particular level cannot absorb or emit a photon at that very frequency!

Linearization of rate equations

In order to solve the linearized rate equations we need to compute these derivatives:

$$\frac{\partial}{\partial n_e}, \frac{\partial}{\partial T}, \frac{\partial}{\partial n_j}, \frac{\partial}{\partial S_k} \quad \text{with respect to} \quad \underline{\underline{A}}, \underline{b}, S_k$$

All derivatives can be computed analytically!

Increases accuracy and stability of numerical solution. More details later.

LTE or NLTE?

When do departures from LTE become important?

LTE is a good approximation, if:

1) Collisional rates dominate for all transitions

$$R_{ij} << C_{ij} \rightarrow P_{ij} = R_{ij} + C_{ij} \approx C_{ij}$$

$$\text{because} \quad \frac{C_{ij}}{C_{ji}} = \left(\frac{n_i}{n_j}\right)^*$$

solution of rate equations $\rightarrow$ LTE

2) $J_\nu = B_\nu$ is a good approximation at all frequencies

$$R_{ij} = R_{ji}$$

$$\rightarrow \frac{n_i}{n_j} = \left(\frac{n_i}{n_j}\right)^*$$

solution of rate equations $\rightarrow$ LTE

LTE or NLTE?

When do departures from LTE become important?

LTE is a bad approximation, if:

1) Collisional rates are small $\quad C_{ij} \sim n_e / \sqrt{T} \qquad n_e \downarrow, T \uparrow \;\Rightarrow\; C_{ij} \downarrow$
2) Radiative rates are large $\quad R_{ij} \sim T^\alpha,\ \alpha > 1 \qquad T \uparrow \;\Rightarrow\; R_{ij} \uparrow$
3) Mean free path of photons is larger than that of electrons
   Example: pure hydrogen plasma

$\Delta z \sim 1/n_H$ (density of neutral H)

Saha: $n_H \sim n_e n_p T^{-3/2} e^{\Delta E/kT} \;\rightarrow\; \Delta z \sim \dfrac{T^{3/2}}{n_e n_p} e^{-\Delta E/kT}$

$n_e \downarrow, T \uparrow \;\Rightarrow\; \Delta z \uparrow$

Departures from LTE occur, if temperatures are high and densities are low

LTE or NLTE?

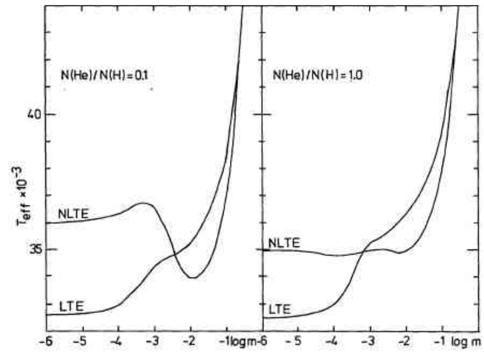


Fig. 4. Temperature stratification in NLTE and LTE for $T_{eff}=45000$ K, $\log g=5$ and two different helium abundances (N(He)/N(H)=0.1 and 1.0)

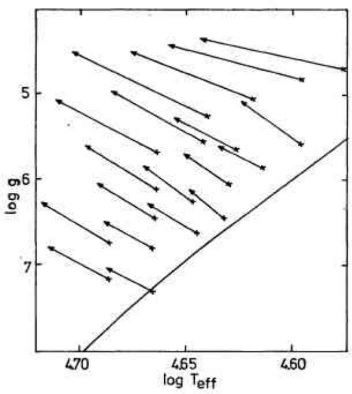


Fig. 8. "Non-LTE vectors" [Displacement due to non-LTE effects in the ($\log g$, $\log T_{eff}$)-diagram] for N(He)/N(H)=0.1

LTE or NLTE?

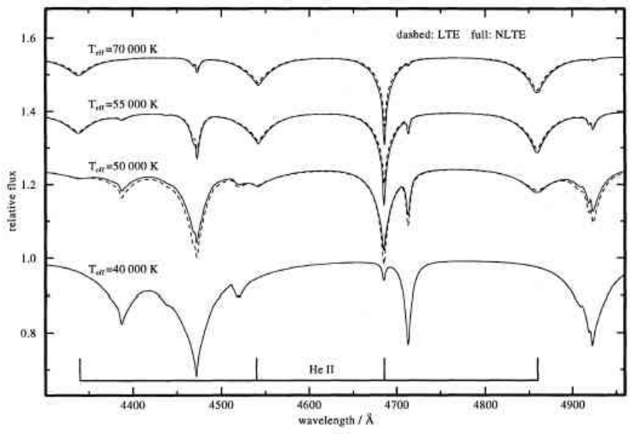
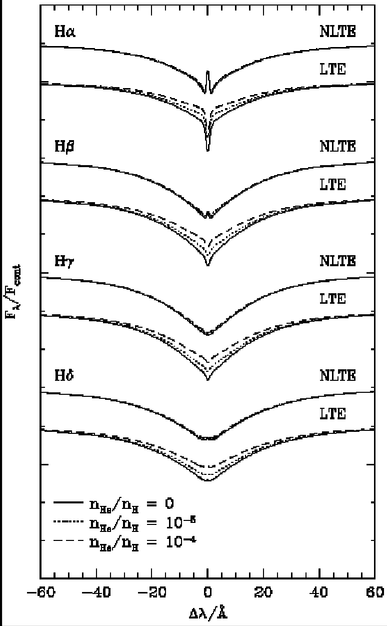
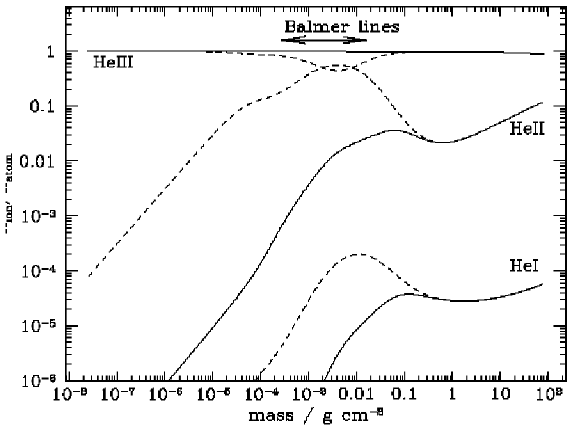


Fig. 3. NLTE effects on synthetic DO line profiles (convoluted by a 2 Å Gauss profile) at various effective temperatures. The two hotter models are calculated at log g=7.5, the cooler ones at log g=8.0. Significant deviations between LTE (dashed) and NLTE spectra occur down to T_{eff}=50 000 K. At T_{eff}=40 000 K the NLTE effects disappear completely, justifying LTE analyses for DB white dwarfs

LTE or NLTE?

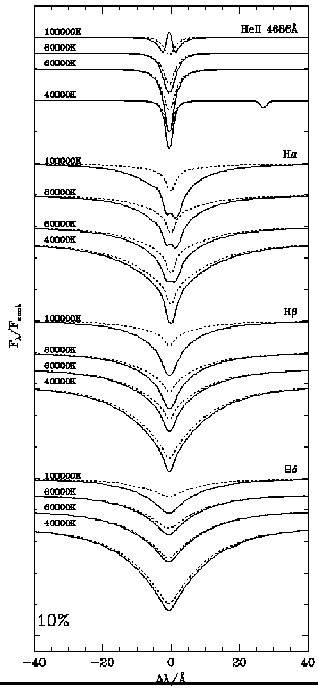
DA white dwarf, T_{eff}= 60000K, log g= 7.5



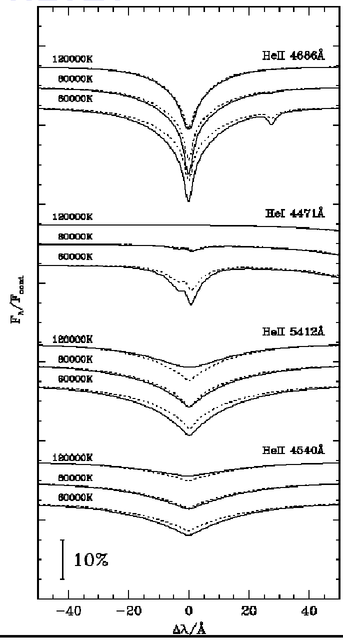


LTE or NLTE?

DAO with log g= 6.5



DO with log g= 7.5



Summary: non-LTE Rate Equations

Complete rate equations

For each atomic level ***i*** of each ion, of each chemical element we have:

$$n_i \sum_{j \neq i} P_{ij} - \sum_{j \neq i} n_j P_{ji} = 0$$

In detail:

$$
\begin{aligned}
& n_i \left[\sum_{j>i} \left(R_{ij} + C_{ij} \right) \right. && \uparrow \text{ excitation and ionization} \\
& \qquad \left. + \sum_{j<i} \left(\frac{n_j}{n_i} \right)^{*} \left(R_{ij} + C_{ji} \right) \right] && \text{rates out of } i \quad \downarrow \text{ de-excitation and recombination} \\
& - \sum_{j>i} n_j \left(\frac{n_i}{n_j} \right)^{*} \left(R_{ji} + C_{ij} \right) && \downarrow \text{ de-excitation and recombination} \\
& - \sum_{j<i} n_j \left(R_{ji} + C_{ji} \right) && \text{rates into } i \quad \uparrow \text{ excitation and ionization} \\
& = 0
\end{aligned}
$$

Solution Strategies

All equations

Radiation Transport	$I_\nu(z), J_\nu(z), H_\nu(z), K_\nu(z)$
Energy Balance	$T(z)$
Hydrostatic Equilibrium	$n_e(z)$
Saha-Boltzmann / Statistical Equilibrium	$n_{ijk}(z)$

Huge system with coupling over depth (RT) and frequency (SE)

Complete Linearisation (Auer Mihalas 1969)
Separate in sub-problems

RT: Short characteristic method

Olson & Kunasz, 1987, JQSRT 38, 325

$$I^+(\tau,\mu,\nu) = I^+(\tau_{\max},\mu,\nu)\exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right) + \int_{\tau}^{\tau_{\max}} S\,(\tau')\exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu}$$

$$I^-(\tau,\mu,\nu) = I^-(0,\mu,\nu)\quad\exp\left(-\frac{\tau}{|\mu|}\right)\qquad + \int_{0}^{\tau} S\,(\tau')\exp\left(-\frac{\tau-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

Solution on a discrete depth grid $\tau_i,\ i=1,ND$ with boundary conditions:

$$I_1^-(\mu,\nu) = I^-(0,\mu,\nu)$$

$$I_{ND}^+(\mu,\nu) = I^+(\tau_{\max},\mu,\nu)$$

Solution along rays passing through whole plane-parallel slab

Short characteristic method

Rewrite with previous depth point as boundary condition for the next interval:

$$I^+(\tau_i,\mu,\nu) = I^+(\tau_{i+1},\mu,\nu)\exp(-\Delta\tau_i) + \Delta I_i^+(S,\mu,\nu)$$

$$I^-(\tau_i,\mu,\nu) = I^-(\tau_{i-1},\mu,\nu)\exp(-\Delta\tau_{i-1}) + \Delta I_i^-(S,\mu,\nu)$$

with

$$\Delta\tau_{i-1} = \frac{(\tau_i - \tau_{i-1})}{|\mu|}$$

using a linear interpolation for the spatial variation of S

the intergrals $\Delta I_i^{\pm}$ can be evaluated as

$$\Delta I_i^{\pm} = \alpha_i^{\pm} S_{i-1} + \beta_i^{\pm} S_i + \gamma_i^{\pm} S_{i+1}$$

Short characteristic method

Out-going rays:

$$\Delta I_i^+(S,\mu,\nu)=\int_{\tau_i}^{\tau_{i+1}} S\exp\left(-\frac{\tau'-\tau_i}{\mu}\right)\frac{d\tau'}{\mu}=\exp\left(\frac{\tau_i}{\mu}\right)\int_{\tau_i}^{\tau_{i+1}} S\exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu}$$

$$x=\frac{\tau'}{\mu}\ ,\ g(x)=\exp(-x)\ ,\ a=\tau_i\ ,\ b=\tau_{i+1}\ ,\ \Delta=\frac{\Delta\tau_i}{\mu}$$

$$\Rightarrow \beta_i^+=w_a=e^{a/\mu}\left(e^{-a/\mu}+\frac{1}{\Delta}\left(e^{-b/\mu}-e^{-a/\mu}\right)\right)=1+\frac{e^{-\Delta}-1}{\Delta}$$

$$\Rightarrow \gamma_i^+=w_b=e^{a/\mu}\left(-e^{-b/\mu}-\frac{1}{\Delta}\left(e^{-b/\mu}-e^{-a/\mu}\right)\right)=-e^{-\Delta}-\frac{e^{-\Delta}-1}{\Delta}$$

Short characteristic method

In-going rays:

$$\Delta I_i^-(S,\mu,\nu)=\int_{\tau_{i-1}}^{\tau_i} S\exp\left(-\frac{\tau_i-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}=\exp\left(-\frac{\tau_i}{|\mu|}\right)\int_{\tau_{i-1}}^{\tau_i} S\exp\left(\frac{\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

$$x=\frac{\tau'}{|\mu|}\ ,\ g(x)=\exp(x)\ ,\ a=\tau_{i-1}\ ,\ b=\tau_i\ ,\ \Delta=\frac{\Delta\tau_{i-1}}{|\mu|}$$

$$\Rightarrow \alpha_i^-=w_a=e^{-b/|\mu|}\left(-e^{a/|\mu|}+\frac{1}{\Delta}\left(e^{b/|\mu|}-e^{a/|\mu|}\right)\right)=-e^{-\Delta}+\frac{1-e^{-\Delta}}{\Delta}$$

$$\Rightarrow \beta_i^-=w_b=e^{-b/|\mu|}\left(e^{b/|\mu|}-\frac{1}{\Delta}\left(e^{b/|\mu|}-e^{a/|\mu|}\right)\right)=1-\frac{1-e^{-\Delta}}{\Delta}$$

Short characteristic method

Also possible: Parabolic instead of linear interpolation

Problem: Scattering $\kappa_e = n_e \sigma_e \;\;,\;\; \eta_e = \kappa_e J = \kappa_e \frac{1}{2}\int_{-1}^{1} I(\mu)d\mu$

Requires iteration

Solution as boundary-value problem
Feautrier scheme

Radiation transfer equation along a ray:

$$\pm\frac{dI_\nu^\pm(\tau)}{d\tau} = I_\nu^\pm(\tau) - S_\nu(\tau)$$

$$\text{pp: } d\tau = \kappa\frac{dt}{d\mu}$$

$$\text{sp: } d\tau = -\kappa dZ$$

Two differential equations for inbound and outbound rays

Definitions by Feautrier (1964):

$u = \frac{1}{2}\left(I^+ + I^-\right)$ symmetric, intensity-like

$v = \frac{1}{2}\left(I^+ - I^-\right)$ antisymmetric, flux-like

Feautrier scheme

Addition and subtraction of both DEQs:

$$\frac{dv(\tau)}{d\tau} = u(\tau) - S_\nu(\tau) \qquad (1)$$

$$\frac{du(\tau)}{d\tau} = v(\tau) \qquad (2)$$

$$\Rightarrow \frac{d^2u(\tau)}{d\tau^2} = u(\tau) - S_\nu(\tau)$$

One DEQ of second order instead of two DEQ of first order

Feautrier scheme

Boundary conditions (pp-case)

Outer boundary

$$I^-(\tau=0) = 0 \rightarrow u(\tau=0) = v(\tau=0)$$

$$(2) \quad \Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=0} = u(\tau=0)$$

... with irradiation

$$I^-(\tau=0) = I_0^- \rightarrow u - v = I^-$$

$$\Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=0} = u(\tau=0) - I_0^-$$

Inner boundary

$$I^+(\tau=\tau_{\max}) = I^+_{\tau_{\max}} \rightarrow u(\tau_{\max}) + v(\tau_{\max}) = I^+_{\tau_{\max}}$$

$$(2) \quad \Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=\tau_{\max}} = I^+_{\tau_{\max}} - u(\tau_{\max})$$

Schuster boundary-value problem

Finite differences

Approximation of the derivatives by finite differences:

$$u-\frac{d^2u}{d\tau^2}=S \quad \text{discretization on a } \tau-\text{scale}$$

first derivative at intermediate points:

$$\tau_{i+1/2}=\frac{1}{2}(\tau_{i+1}+\tau_i)$$

$$\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i+1/2}}\approx\frac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i} \quad \text{and} \quad \left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i-1/2}}\approx\frac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}$$

second derivative:

$$\left.\frac{d}{d\tau}\left(\frac{du(\tau)}{d\tau}\right)\right|_{\tau_i}=\frac{\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i+1/2}}-\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i-1/2}}}{\tau_{i+1/2}-\tau_{i-1/2}}$$

$$\left.\frac{du^2(\tau)}{d\tau^2}\right|_{\tau_i}\approx\frac{\frac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i}-\frac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}}{\frac{1}{2}(\tau_{i+1}-\tau_{i-1})}$$

Stellar Atmospheres: Solution Strategies

Finite differences

Approximation of the derivatives by finite differences:

$$u-\frac{d^2u}{d\tau^2}=S \quad \text{discretisation on a } \tau-\text{scale}$$

$$\Rightarrow u_i-\frac{\frac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i}-\frac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}}{\frac{1}{2}(\tau_{i+1}-\tau_{i-1})}=S_i \quad , \quad i=2\cdots ND-1$$

$$\boxed{\begin{aligned}&\Rightarrow -A_iu_{i-1}+B_iu_i-C_iu_{i+1}=S_i \quad , \quad i=2\cdots ND-1\\ &A_i=\left[\frac{1}{2}(\tau_i-\tau_{i-1})(\tau_{i+1}-\tau_{i-1})\right]^{-1}\\ &C_i=\left[\frac{1}{2}(\tau_{i+1}-\tau_i)(\tau_{i+1}-\tau_{i-1})\right]^{-1}\\ &B_i=1+A_i+C_i\end{aligned}}$$

Linear system of equations

$$\begin{bmatrix} B_1 & -C_1 & & & 0 \\ -A_2 & B_2 & -C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & -C_{ND-1} \\ & & & -A_{ND} & B_{ND} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{ND-1} \\ u_{ND} \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ \vdots \\ W_{ND-1} \\ W_{ND} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ \vdots \\ S_{ND-1} \\ S^*_{ND} \end{bmatrix}$$

Linear system for u_i

Use Gauss-Jordan elimination for solution

Upper diagonal matrix

1st step:

$$\begin{bmatrix} 1 & -\tilde{C}_1 & & & 0 \\ & 1 & -\tilde{C}_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ 0 & & & \ddots & -\tilde{C}_{ND-1} \\ & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{ND-1} \\ u_{ND} \end{bmatrix} = \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \vdots \\ \vdots \\ \tilde{W}_{ND-1} \\ \tilde{W}_{ND} \end{bmatrix}$$

$$\boxed{\begin{array}{lll} i=1 & \tilde{C}_1 = B_1^{-1} C_1 & \tilde{W}_1 = B_1^{-1} W_1 \\ i = 2 \cdots ND & \tilde{C}_i = \left(B_i - A_i \tilde{C}_{i-1}\right)^{-1} C_i & \tilde{W}_i = \left(B_i - A_i \tilde{C}_{i-1}\right)^{-1} \left(W_i + A_i \tilde{W}_{i-1}\right) \end{array}}$$

Back-substitution

2nd step:

$$\boxed{\begin{array}{ll} i = ND & u_{ND} = \tilde{W}_{ND} \\ i = ND-1\cdots 1 & u_i = \tilde{W}_i + \tilde{C}_i u_{i+1} \end{array}}$$

Solution fulfils differential equation as well as both boundary conditions

Remark: for later generalization the matrix elements are treated as matrices (non-commutative)

Complete Linearization

Auer & Mihalas 1969

Newton-Raphson method in $\mathbb{R}^n$

Solution according to Feautrier scheme

Unknown variables:

$$\vec{\psi}_i = \begin{bmatrix} \vec{J}_i \\ \vec{n}_i \end{bmatrix} \quad , \; i = 1\cdots ND \qquad \psi = [\vec{\psi}_1, \cdots, \vec{\psi}_i, \cdots, \vec{\psi}_{ND}]^T$$

Equations:

$$-A_{i,k} J_{i-1,k} + B_{i,k} J_{i,k} - C_{i,k} J_{i+1,k} - S_{i,k}(\vec{n}_i) = 0 \qquad \text{NF transfer equations}$$

$$P(\vec{J}_i)\vec{n}_i - \vec{b}_i = 0 \qquad \text{NL equations for SE}$$

System of the form:

$$f_{i,\alpha}(\psi) = 0 \quad , \quad \alpha = 1\cdots NF + NL$$

Complete Linearization

Start approximation: $f_{i,\alpha}(\psi^0) \neq 0$

Now looking for a correction so that

$$f_{i,\alpha}(\psi^0 + \delta\psi) = 0 \quad \forall\ i,\alpha$$

Taylor series:

$$0 = f_{i,\alpha}(\psi) = f_{i,\alpha}(\psi^0 + \delta\psi)$$

$$= f_{i,\alpha}(\psi^0) + \sum_{i=1}^{ND} \left\{ \sum_{k=1}^{NF} \frac{\partial f_{i,\alpha}}{\partial J_{i,k}} \delta J_{i,k} + \sum_{l=1}^{NL} \frac{\partial f_{i,\alpha}}{\partial n_{i,l}} \delta n_{i,l} \right\} \Bigg|_{\psi^0} + \cdots$$

Linear system of equations for ND(NF+NL) unknowns $\delta J_{i,k}$, $\delta n_{i,l}$

Converges towards correct solution

Many coefficients vanish

Complete Linearization - structure

Only neighbouring depth points (2nd order transfer equation with tri-diagonal depth structure and diagonal statistical equations): $f_{i,\alpha}(\psi) = f_{i,\alpha}(\vec{\psi}_{i-1}, \vec{\psi}_i, \vec{\psi}_{i+1})$

Results in tri-diagonal block scheme (like Feautrier)

$$-A_i \delta\vec{\psi}_{i-1} + B_i \delta\vec{\psi}_i - C_i \delta\vec{\psi}_{i+1} = \vec{L}_i$$

$$-\left(\begin{array}{ccc|c} \ddots & & 0 & \\ & A_{i,k} & & 0 \\ 0 & & \ddots & \\ \hline & 0 & & 0 \end{array}\right) \left[\begin{array}{c} \delta\vec{J}_{i-1} \\ \hline \delta\vec{n}_{i-1} \end{array}\right] + \left(\begin{array}{ccc|ccc} \ddots & 0 & & \square & \square & \square \\ & B_{i,k} & & \square & \square & \square \\ 0 & & \ddots & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array}\right) \left[\begin{array}{c} \delta\vec{J}_i \\ \hline \delta\vec{n}_i \end{array}\right]$$

$$-\left(\begin{array}{ccc|c} \ddots & & 0 & \\ & C_{i,k} & & 0 \\ 0 & & \ddots & \\ \hline & 0 & & 0 \end{array}\right) \left[\begin{array}{c} \delta\vec{J}_{i+1} \\ \hline \delta\vec{n}_{i+1} \end{array}\right] = -\left[\begin{array}{c} \square \\ \square \\ f_{i,\alpha}(\psi^0) \\ \square \\ \square \\ \square \end{array}\right]$$

Complete Linearization - structure

Transfer equations: coupling of $J_{i-1,k}$, $J_{i,k}$, and $J_{i+1,k}$ at the same frequency point:

$\rightarrow$ Upper left quadrants of A_i, B_i, C_i describe 2nd derivative $\dfrac{d^2 J}{d\tau^2}$

Source function is local:

$\rightarrow$ Upper right quadrants of A_i, C_i vanish

Statistical equations are local

$\rightarrow$ Lower right and lower left quadrants of A_i, C_i vanish

Complete Linearization - structure

Matrix $B_{i:}$

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} 1 & \quad\ldots\quad & NF & \mid & 1 & \quad\ldots\quad & NL \end{array} \\
B_i = & \left(\begin{array}{ccc|ccc}
\ddots & & 0 & & \vdots & \\
 & B_{i,k} & & \cdots & -\dfrac{\partial S_{i,k}}{\partial n_{i,l'}} & \cdots \\
0 & & \ddots & & \vdots & \\
\hline
 & \vdots & & & \vdots & \\
\cdots & \displaystyle\sum_{m=1}^{NL} \frac{\partial (P_i)_{l,m}}{\partial J_{i,k'}} n_{i,m} & \cdots & \cdots & (P_i)_{l,l'} & \cdots \\
 & \vdots & & & \vdots &
\end{array}\right)
\begin{array}{c} 1 \\ \vdots \\ NF \\ 1 \\ \vdots \\ NL \end{array}
\end{array}
$$

Complete Linearization

Alternative (recommended by Mihalas): solve SE first and linearize afterwards: $P(\vec{J}_i)\vec{n}_i - \vec{b}_i = 0 \;\rightarrow \vec{n}_i = P(\vec{J}_i)^{-1}\vec{b}_i$

Newton-Raphson method:

- Converges towards correct solution
- Limited convergence radius
- In principle quadratic convergence, however, not achieved because variable Eddington factors and τ-scale are fixed during iteration step
- CPU~ND (NF+NL)^3 $\rightarrow$ simple model atoms only
  – Rybicki scheme is no relief since statistical equilibrium not as simple as scattering integral

Energy Balance

$\rightarrow$ Including radiative equilibrium into solution of radiative transfer $\rightarrow$ Complete Linearization for model atmospheres

$\rightarrow$ Separate solution via temperature correction

+ Quite simple implementation
+ Application within an iteration scheme allows completely linear system $\rightarrow$ next chapter
– No direct coupling
– Moderate convergence properties

Temperature correction – basic scheme

0. start approximation for $T(\tau) \leftarrow T_0(\tau)$
1. formal solution $J_\nu = \Lambda_\nu S_\nu(T)$
2. correction $T(\tau) \leftarrow T(\tau) + \Delta T(\tau)$
3. convergence?

Several possibilities for step 2 based on radiative equilibrium or flux conservation

Generalization to non-LTE not straightforward

With additional equations towards full model atmospheres:

- Hydrostatic equilibrium
- Statistical equilibrium

LTE

Strict LTE $S_\nu(\tau) = B_\nu(T(\tau))$

Scattering $S_\nu(\tau) = (1-\beta_e)B_\nu(T(\tau)) + \beta_e J_\nu(\tau)$

Simple correction from radiative equilibrium:

$$\int_{\nu=0}^{\infty} \kappa(\tau,\nu)\left(J_\nu(\tau,\nu) - B_\nu(T(\tau),\nu)\right) d\nu \neq 0$$

$$\xrightarrow[\Delta T]{} \int_{\nu=0}^{\infty} \kappa(\tau,\nu)\left(J_\nu(\tau,\nu) - B_\nu\left[T(\tau) + \Delta T(\tau)\right]\right) d\nu = 0$$

$$\Rightarrow \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu - \Delta T \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)} \right) d\nu = 0$$

$$\Rightarrow \Delta T = \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu\right) d\nu \Bigg/ \int_{\nu=0}^{\infty} \kappa \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)} d\nu$$

LTE

Problem:

$$\Delta T = \int_{\nu=0}^{\infty} \kappa \left(J_\nu - B_\nu\right) d\nu \Big/ \int_{\nu=0}^{\infty} \kappa \frac{\partial B_\nu}{\partial T}\Big|_{T=T(\tau)} d\nu$$

$$J_\nu \xrightarrow[\tau\to\infty]{} B_\nu \quad \text{independent of the temperature } \Rightarrow \Delta T \to 0$$

Gray opacity (κ independent of frequency):

$$\int_{\nu=0}^{\infty} \kappa(\nu)\left(J_\nu - B_\nu\right) d\nu \to \kappa(J-B)$$

$$\to \kappa(J - B - \Delta B) = 0$$

$$\to \kappa(J-B) = \kappa\Delta B$$

$$\xrightarrow[\text{0.Moment equation}]{} \frac{dH}{dt} = \kappa\Delta B$$

deviation from constant flux provides temperature correction

Unsöld-Lucy correction

Unsöld (1955) for gray LTE atmospheres, generalized by Lucy (1964) for non-gray LTE atmospheres

0-th moment: $\frac{dH_\nu}{dt} = \kappa_\nu \left(J_\nu - B_\nu\right)$

$$\int \cdots d\nu \to \frac{dH}{d\tau} = \frac{\kappa_J}{\kappa_B} J - B \;,\; \kappa_B B = \int_{\nu=0}^{\infty} \kappa_\nu B_\nu d\nu \;,\; \kappa_J J = \int_{\nu=0}^{\infty} \kappa_\nu J_\nu d\nu \;,\; d\tau = \kappa_B dt$$

1st moment: $\frac{dK_\nu}{dt} = \kappa_\nu H_\nu$

$$\int \cdots d\nu \to \frac{dK}{d\tau} = \frac{\kappa_H}{\kappa_B} H \;,\; \kappa_H H = \int_{\nu=0}^{\infty} \kappa_\nu H_\nu d\nu$$

now new quantities J', H', K' fulfilling radiative equilibrium (local) and flux conservation (non local)

radiative equilibrium: $\frac{dH'}{d\tau} = \frac{\kappa_J}{\kappa_B} J' - B' = 0$

flux conservation: $\frac{dK'}{d\tau} = \frac{\kappa_H}{\kappa_B} H' = \frac{\kappa_H}{\kappa_B} \frac{\sigma}{4\pi} T_{\text{eff}}^4$

Unsöld-Lucy correction

Now corrections to obtain new quantities:

$$\Delta X = X' - X$$

$$\frac{d\Delta K}{d\tau} = \frac{\kappa_H}{\kappa_B}\Delta H \quad \text{integrate} \quad \rightarrow \Delta K = \Delta K(0) + \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'$$

$$K = \int_0^\infty K_\nu d\nu = \int_0^\infty f_\nu J_\nu d\nu = fJ \;\; , \;\; H(0) = \int_0^\infty H_\nu(0) d\nu = \int_0^\infty h_\nu J_\nu(0) d\nu = hJ(0)$$

$$\rightarrow \Delta K = \frac{f(0)\Delta H(0)}{h} + \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau' = f\Delta J$$

$$\frac{d\Delta H}{d\tau} = \frac{\kappa_J}{\kappa_B}\Delta J - \Delta B \rightarrow \Delta B = -\frac{d\Delta H}{d\tau} + \frac{\kappa_J}{\kappa_B}\left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f}\int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'\right)$$

$$\Delta B = \frac{4\sigma T^3}{\pi}\Delta T = -\frac{dH'}{d\tau} + \frac{dH}{d\tau} + \frac{\kappa_J}{\kappa_B}\left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f}\int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'\right)$$

$$\boxed{\Delta T = \frac{\pi}{4\sigma T^3}\left[\frac{\kappa_J}{\kappa_B}J - B + \frac{\kappa_J}{\kappa_B}\left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f}\int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'\right)\right]}$$

Unsöld-Lucy correction

$$\boxed{\Delta T = \frac{\pi}{4\sigma T^3}\left[\underbrace{\frac{\kappa_J}{\kappa_B}J - B}_{} + \underbrace{\frac{\kappa_J}{\kappa_B}\left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f}\int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B}\Delta H d\tau'\right)}_{}\right]}$$

„Radiative equilibrium“ part good at small optical depths but poor at large optical depths $J \rightarrow B$

„Flux conservation“ part good at large optical depths but poor at small optical depths $\frac{dH}{d\tau} \rightarrow 0$

Unsöld-Lucy scheme typically requires damping

Still problems with strong resonance lines, i.e. radiative equilibrium term is dominated by few optically thick frequencies

NLTE Model Atmospheres

Radiation Transport and Sattistical Equilibrium are very closely coupled

Simple separation (Lamda Iteration) does not work

Complete Linearization

Accelerated Lambda Iteration

Lambda Iteration

Split RT and SE+RE:

$$J^{new} = \Lambda S^{old}(n,T)$$ RT formal solution

$$\underline{\underline{A(J,T)}}\,\underline{n}^{new} = \underline{b}$$ SE

$$\int_0^\infty \kappa(\nu,n,T)\left(J_\nu - S_\nu(\nu,n,T)\right)d\nu = 0$$ RE

- Good: SE is linear (if a separate T-correction scheme is used)
- Bad: SE contain old values of n,T (in rate matrix A)

Disadvantage: not converging, this is a Lambda iteration!

Accelerated Lambda Iteration (ALI)

Again: split RT and SE+RE but now use ALI

$$J^{new} = \Lambda S^{old}(n^{old},T^{old}) + \Lambda^* S^{new}(n^{new},T^{new}) - \Lambda^* S^{old}(n^{old},T^{old}) \qquad \text{RT}$$

$$\underline{\underline{A}}(J^{new},T^{new})\underline{n}^{new} = \underline{b} \qquad \text{SE}$$

$$\int_0^\infty \kappa(\nu,n^{new},T^{new})\left(J_\nu^{new} - S_\nu(\nu,n^{new},T^{new})\right)d\nu = 0 \qquad \text{RE}$$

- Good: SE contains new quantities n, T
- Bad: Non-Linear equations → linearization (but without RT)

Basic advantage over Lambda Iteration: ALI converges!

Example: ALI working on Thomson scattering problem

$S = (1-\beta_e)B + \beta_e J$ source function with scattering, problem: J unknown→iterate

$$\Rightarrow J^{new} = (\Lambda - \Lambda^*)S^{old} + \Lambda^* S^{new}$$

$$= (\Lambda - \Lambda^*)S^{old} + \Lambda^*\left((1-\beta_e)B^{new} + \beta_e J^{new}\right) \qquad J^{FS}\text{:=formal solution on } S^{old}$$

$$= J^{FS} - \Lambda^*\left((1-\beta_e)B^{old} + \beta_e J^{old} - (1-\beta_e)B^{new} - \beta_e J^{new}\right) \qquad B^{old} = B^{new}$$

$$= J^{FS} - \Lambda^*\left(\beta_e J^{old} - \beta_e J^{new}\right) \qquad \text{solve for } J^{new}$$

$$\Rightarrow J^{new} = \left[1 - \Lambda^*\beta_e\right]^{-1}\left(J^{FS} - \Lambda^*\beta_e J^{old}\right) \qquad \text{subtract } J^{old} \text{ on both sides}$$

$$\Rightarrow J^{new} - J^{old} = \left[1 - \Lambda^*\beta_e\right]^{-1}\left(J^{FS} - J^{old}\right)$$

amplification factor

Interpretation: iteration is driven by difference (J^{FS}-J^{old}) but: this difference is amplified, hence, iteration is accelerated.
Example: β_e=0.99; at large optical depth Λ^* almost 1 → strong amplification

What is a good Λ*?

The choice of Λ* is in principle irrelevant but in practice it decides about the success/failure of the iteration scheme.

First (useful) Λ* (Werner & Husfeld 1985):

$$\Lambda_\nu^*(\tau,\tau')\,S_\nu(\tau') = \begin{cases} S_\nu(\tau) & \tau > \gamma \\ 0 & \tau \le \gamma \end{cases}$$

A few other, more elaborate suggestions until Olson & Kunasz (1987): Best Λ* is the diagonal of the Λ-matrix

(Λ-matrix is the numerical representation of the integral operator Λ)

We therefore need an efficient method to calculate the elements of the Λ-matrix (are essentially functions of τ_ν).

Could compute directly elements representing the Λ-integral operator, but too expensive (E_1 functions). Instead: use solution method for transfer equation in differential (not integral) form: short characteristics method

Towards a linear scheme

Λ* acts on S, which makes the equations non-linear in the occupation numbers

- Idea of Rybicki & Hummer (1992): use $J=\Delta J+\Psi^*\eta^{new}$ instead
- Modify the rate equations slightly:

$$R_{ij}n_i = 4\pi\int_0^\infty \frac{\sigma_{ij}}{h\nu} n_i J_\nu d\nu = 4\pi\int_0^\infty \frac{\sigma_{ij}}{h\nu} n_i \left(\Psi^*\eta(n)+\Delta J\right)d\nu$$

$$R_{ji}n_j = 4\pi\left(\frac{n_i}{n_j}\right)^*\int_0^\infty \frac{\sigma_{ij}}{h\nu} n_j \left(J_\nu + \frac{2h\nu^3}{c^2}\right)d\nu$$

$$= 4\pi\left(\frac{n_i}{n_j}\right)^*\int_0^\infty \frac{\sigma_{ij}}{h\nu} n_j \left(\Psi^*\eta(n)+\Delta J + \frac{2h\nu^3}{c^2}\right)d\nu$$

Stellar Atmospheres

This was the contents of our lecture:

Radiation field
Radiation transfer
Emission and absorption
Energy balance and Radiative equilibrium
Hydrostatic equilibrium
Solution Strategies for Stellar atmosphere models

Stellar Atmospheres

This was the contents of our lecture:

Radiation field
Radiation transfer
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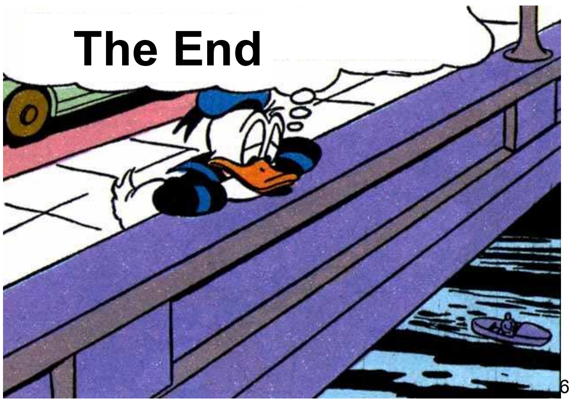
Radiative equilibrium
Hydrostatic equilibrium
Stellar atmosphere models

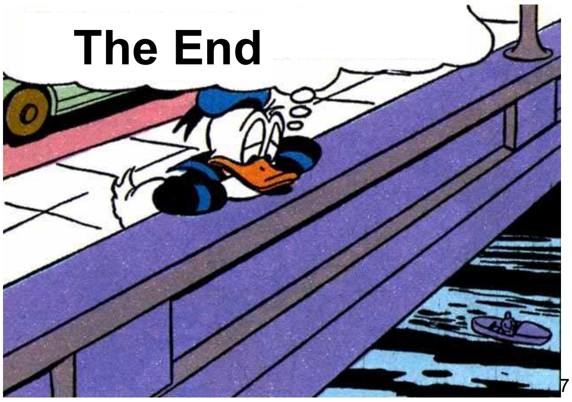


Stellar Atmospheres

This was the contents of our lecture:

Radiation field
Radiation transfer
Emission and absorption
Radiative equilibrium
Hydrostatic equilibrium
Stellar atmosphere models

Thank you for listening !



Stellar Atmospheres in Non-LTE

Stellar Atmospheres in Non-LTE

radiation transfer
radiative equilibrium
hydrostatic equation

$\left.\begin{array}{l} \text{radiation transfer} \\ \text{radiative equilibrium} \\ \text{hydrostatic equation} \end{array}\right\}$ solve consistently (*) $\begin{array}{l} I_\nu \text{ or } J_\nu \\ T \\ N \end{array}$

Parameters: T_{eff}, $log\ g$, y_k

LTE: $S_\nu = B_\nu$ strict LTE

$S_\nu = \rho J_\nu + (1-\rho) B_\nu$ including scattering

Population numbers by Saha-Boltzmann equations

NLTE: $\frac{dn_i}{dt} = 0 \rightarrow \underline{\underline{A}}\,\underline{n} = \underline{b}$ rate equations

Solution Methods

(∗) is a non-linear system of equations, we look for the solution vector:

$$\boxed{\underline{\psi}^d = \left(n_1, \cdots, n_{NL}, N, T, n_e, J_{\nu_1}, \cdots, J_{\nu_{NF}}\right)^d}$$ ***d***=depth index

with $\underline{\underline{M}}^d\left(\underline{\psi}\right)\underline{\psi}^d = \underline{c}^d\left(\underline{\psi}\right)$

Solution principle: Newton-Raphson iteration

Solution methods:

1. Complete Linearization (Auer & Mihalas 1969)
2. Multi-frequency/multi-gray (Anderson 1985)
3. ALI method (Werner, Husfeld 1985, 1986)

All methods have in common: linearization and iteration

For that, it is necessary to invert matrices (Jacobi matrix) with rank = number of equations

Numerical limit: matrix inversion limits rank to the order of ~ 100

Stellar Atmospheres: Non-LTE Stellar Atmospheres

Complete Linearization

Linearizes all equations

Enabled break-through for first calculation of NLTE models, quite robust method

Depth coupling by radiation transfer

⇒ Feautrier scheme

Disadvantage: Capacity limit quickly reached

e.g. model atom with hydrogen and helium:

$$\left.\begin{array}{l}\text{20 NLTE levels}\\\text{80 frequency points}\end{array}\right\}\ \text{100 equations}$$

(5 for each spectral line, 2 for each bound-free edge)

Only rather rudimentary representation of plasma

⇒ Number of equations must be reduced

Anderson's method

Does not linearize the transfer equation with respect to all frequency points. First: grouping of frequency points in energy blocks. Then: linearization of these quantities.

Number of blocks determines the dimension of the system of equations.

In some sense related to multi-grid methods.

Very clever method, BUT: requires physical motivation for grouping of frequencies. Must be done manually, quite cumbersome, much experience and physical insight by user necessary. Was essentially used by inventor himself, is not used any more.

ALI method

Accelerated Lambda Iteration

Eliminates the explicit inclusion of the transfer equation into the linearization scheme by using instead an implicit approximate solution for J_ν:

Lambda iteration:

i=iteration counter

$$J_\nu^{(i)} = \Lambda S_\nu^{(i-1)}\left(\underline{n}^{(i-1)}\right)$$

$$\underline{\underline{A}}\left(J_\nu^{(i)}\right)\underline{n}^{(i)} = \underline{b}$$

ALI:

$$J_\nu^{(i)} = \Lambda S_\nu^{(i-1)}\left(\underline{n}^{(i-1)}\right) + \Lambda^* S_\nu^{(i)}\left(\underline{n}^{(i)}\right) - \Lambda^* S_\nu^{(i-1)}\left(\underline{n}^{(i-1)}\right)$$

$$J_\nu^{(i)} = \Lambda^* S_\nu^{(i)}\left(\underline{n}^{(i)}\right) + \Delta J_\nu^{(i-1)}$$

$$\underline{\underline{A}}\left(J_\nu^{(i)}\right)\underline{n}^{(i)} = \underline{b}$$

ALI method

Advantage: number of frequency points no longer appears in dimension of equation system to be linearized (but calculation of derivatives of η_ν, κ_ν w.r.t. source function)

No explicit depth coupling, i.e. local linearized equations for every depth point

Starting solution $\underline{\psi}^d = (n_1, \cdots, n_{NL}, N, T, n_e)^d$

Calculate correction $\underline{\delta\psi}^d = (\delta n_1, \cdots, \delta n_{NL}, \delta N, \delta T, \delta n_e)^d$

from linearized equation $\underline{\underline{M}}^d \, \underline{\delta\psi}^d = \underline{c}^d$

$$\underline{\delta\psi}^d = \left(\underline{\underline{M}}^d\right)^{-1} \underline{c}^d$$

Improved solution $\underline{\psi}^d + \underline{\delta\psi}^d \rightarrow \underline{\psi}^d$

Radiation Transport as Boundary-Value Problem of Differential Equations

Solution with given source function

- Formal Solution, applications:
  - Strict LTE, $S_\nu = B_\nu(T)$
  - Step within iterative method
- Numerical integration, short characteristics method
- Algebraic equation, Feautrier method

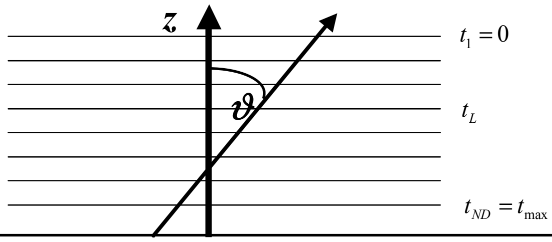
Solution by numerical integration

Emergent intensity, plane-parallel geometry

Depth grid, ND depth points

- Geometrical depth $t_L, L = 1 \cdots ND$
- Optical depth $\tau_L, L = 1 \cdots ND, \tau_L(\nu) = \int_{t'=0}^{t_L} \kappa(\nu, t')dt'$

$$I_\nu(\tau = 0, \mu), \quad \mu = \cos\vartheta$$



Emergent intensity

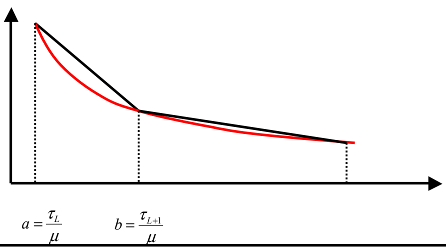
$$\boxed{I_\nu^+(\tau = 0, \mu) = \int_{\tau'=0}^{\tau_{\max}} S_\nu(\tau') \exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu} + I_\nu^+(\tau_{\max})\exp\left(-\frac{\tau_{\max}}{\mu}\right)}$$

Numerical integration: $I_\nu^+(\tau = 0, \mu) = \sum_{L=1}^{ND} S_L w_L$

Trapezoidal rule,
naive approach:

$$I = \left(S_a e^{-a} + S_b e^{-b}\right)\frac{b-a}{2}$$

Not very smart, systematic summation of approximation errors



Proper integration weights

Problem: $$I = \int_a^b f(x)g(x)dx$$

Interval: (a,b)

Integrand: $f(x)$

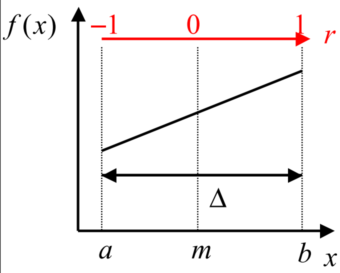
Weight function: $g(x)$

Including weight function in integration weights is smart if:

$f(x)$ is less curved than $f(x) \cdot g(x)$

$g(x)$ and $x \cdot g(x)$ have handy antiderivatives

Determination of proper weights



$$\Delta = b - a$$

$$m = \frac{(b+a)}{2}$$

$$x = m + r\frac{\Delta}{2}$$

$$dx = \frac{\Delta}{2}dr$$

$$f(x) = \frac{f(b)+f(a)}{2} + r\frac{f(b)-f(a)}{2}$$, even and odd

$$I = \int_a^b f(x)g(x)dx = \frac{\Delta}{4}\int_{r=-1}^{1}\left(f(b)+f(a)+r\left(f(b)-f(a)\right)\right)g\left(m+r\Delta/2\right)dr$$

like wise:

$$g(x) = g^e(r) + g^o(r)$$

$$g^e(r) = \frac{1}{2}\left(g\left(m+r\Delta/2\right) + g\left(m-r\Delta/2\right)\right)$$

$$g^o(r) = \frac{1}{2}\left(g\left(m+r\Delta/2\right) - g\left(m-r\Delta/2\right)\right)$$

Determination of proper weights

Integration over symmetric interval leaves only even integrands:

$$I=\frac{\Delta}{4}\left[\left(f(b)+f(a)\right)\int\limits_{r=-1}^{1} g^{e}(r)dr \;+\; \left(f(b)-f(a)\right)\int\limits_{r=-1}^{1} rg^{o}(r)dr\right]$$

$$=\frac{\Delta}{2}\left[\left(f(b)+f(a)\right)\int\limits_{r=0}^{1} g^{e}(r)dr \;+\; \left(f(b)-f(a)\right)\int\limits_{r=0}^{1} rg^{o}(r)dr\right]$$

$$=\frac{\Delta}{2}\left[\left(f(b)+f(a)\right)\mathrm{G} \;+\; \left(f(b)-f(a)\right)H\right]=w_a f(a)+w_b f(b)$$

$$\Rightarrow w_a=\frac{\Delta}{2}(G-H)$$

$$\Rightarrow w_b=\frac{\Delta}{2}(G+H)$$

Examples

$g(x)=x$

$$G=m, H=\frac{\Delta}{6}$$

$$w_a=\frac{\Delta}{2}\left(m-\frac{\Delta}{6}\right)$$

$$w_b=\frac{\Delta}{2}\left(m+\frac{\Delta}{6}\right)$$

$g(x)=x^2$

$$G=m^2+\frac{\Delta^2}{12}, H=m\frac{\Delta}{3}$$

$$w_a=\frac{\Delta}{2}\left(m^2+\frac{\Delta^2}{12}-m\frac{\Delta}{3}\right)$$

$$w_b=\frac{\Delta}{2}\left(m^2+\frac{\Delta^2}{12}+m\frac{\Delta}{3}\right)$$

$g(x)=e^{-x}$

$$G=\frac{1}{\Delta}\left(e^{-a}-e^{-b}\right)$$

$$H=\frac{1}{\Delta}\left(-e^{-a}-e^{-b}+\frac{2}{\Delta}\left(e^{-a}-e^{-b}\right)\right)$$

$$w_a=e^{-a}-\frac{1}{\Delta}\left(e^{-a}-e^{-b}\right)$$

$$w_b=-e^{-b}+\frac{1}{\Delta}\left(e^{-a}-e^{-b}\right)$$

$g(x)=e^{x}$

$$G=\frac{1}{\Delta}\left(-e^{a}+e^{b}\right)$$

$$H=\frac{1}{\Delta}\left(e^{a}+e^{b}+\frac{2}{\Delta}\left(e^{a}-e^{b}\right)\right)$$

$$w_a=-e^{a}-\frac{1}{\Delta}\left(e^{a}-e^{b}\right)$$

$$w_b=e^{-b}+\frac{1}{\Delta}\left(e^{a}-e^{b}\right)$$

Short characteristic method

Olson & Kunasz, 1987, JQSRT 38, 325

$$I^{+}(\tau,\mu,\nu) = I^{+}(\tau_{\max},\mu,\nu)\exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right) + \int_{\tau}^{\tau_{\max}} S\,(\tau')\exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu}$$

$$I^{-}(\tau,\mu,\nu) = I^{-}(0,\mu,\nu) \quad \exp\left(-\frac{\tau}{|\mu|}\right) \quad + \int_{0}^{\tau} S\,(\tau')\exp\left(-\frac{\tau-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

Solution on a discrete depth grid $\tau_i,\ i = 1, ND$ with boundary conditions:

$$I_1^{-}(\mu,\nu) = I^{-}(0,\mu,\nu)$$

$$I_{ND}^{+}(\mu,\nu) = I^{+}(\tau_{\max},\mu,\nu)$$

Solution along rays passing through whole plane-parallel slab

Short characteristic method

Rewrite with previous depth point as boundary condition for the next interval:

$$I^{+}(\tau_i,\mu,\nu) = I^{+}(\tau_{i+1},\mu,\nu)\exp\left(-\Delta\tau_i\right) + \Delta I_i^{+}(S,\mu,\nu)$$

$$I^{-}(\tau_i,\mu,\nu) = I^{-}(\tau_{i-1},\mu,\nu)\exp\left(-\Delta\tau_{i-1}\right) + \Delta I_i^{-}(S,\mu,\nu)$$

with

$$\Delta\tau_{i-1} = \frac{\left(\tau_i - \tau_{i-1}\right)}{|\mu|}$$

using a linear interpolation for the spatial variation of S

the intergrals $\Delta I_i^{\pm}$ can be evaluated as

$$\Delta I_i^{\pm} = \alpha_i^{\pm} S_{i-1} + \beta_i^{\pm} S_i + \gamma_i^{\pm} S_{i+1}$$

Short characteristic method

Out-going rays:

$$\Delta I_i^+(S,\mu,\nu) = \int_{\tau_i}^{\tau_{i+1}} S \exp\left(-\frac{\tau'-\tau_i}{\mu}\right)\frac{d\tau'}{\mu} = \exp\left(\frac{\tau_i}{\mu}\right)\int_{\tau_i}^{\tau_{i+1}} S \exp\left(-\frac{\tau'}{\mu}\right)\frac{d\tau'}{\mu}$$

$$x = \frac{\tau'}{\mu}\ ,\ g(x) = \exp(-x)\ ,\ a = \tau_i\ ,\ b = \tau_{i+1}\ ,\ \Delta = \frac{\Delta\tau_i}{\mu}$$

$$\Rightarrow \beta_i^+ = w_a = e^{a/\mu}\left(e^{-a/\mu} + \frac{1}{\Delta}\left(e^{-b/\mu} - e^{-a/\mu}\right)\right) = 1 + \frac{e^{-\Delta}-1}{\Delta}$$

$$\Rightarrow \gamma_i^+ = w_b = e^{a/\mu}\left(-e^{-b/\mu} - \frac{1}{\Delta}\left(e^{-b/\mu} - e^{-a/\mu}\right)\right) = -e^{-\Delta} - \frac{e^{-\Delta}-1}{\Delta}$$

Short characteristic method

In-going rays:

$$\Delta I_i^-(S,\mu,\nu) = \int_{\tau_i-1}^{\tau_i} S \exp\left(-\frac{\tau_i-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|} = \exp\left(-\frac{\tau_i}{|\mu|}\right)\int_{\tau_{i-1}}^{\tau_i} S \exp\left(\frac{\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

$$x = \frac{\tau'}{|\mu|}\ ,\ g(x) = \exp(x)\ ,\ a = \tau_{i-1}\ ,\ b = \tau_i\ ,\ \Delta = \frac{\Delta\tau_{i-1}}{|\mu|}$$

$$\Rightarrow \alpha_i^- = w_a = e^{-b/|\mu|}\left(-e^{a/|\mu|} + \frac{1}{\Delta}\left(e^{b/|\mu|} - e^{a/|\mu|}\right)\right) = -e^{-\Delta} + \frac{1-e^{-\Delta}}{\Delta}$$

$$\Rightarrow \beta_i^- = w_b = e^{-b/|\mu|}\left(e^{b/|\mu|} - \frac{1}{\Delta}\left(e^{b/|\mu|} - e^{a/|\mu|}\right)\right) = 1 - \frac{1-e^{-\Delta}}{\Delta}$$

Short characteristic method

Also possible: Parabolic instead of linear interpolation

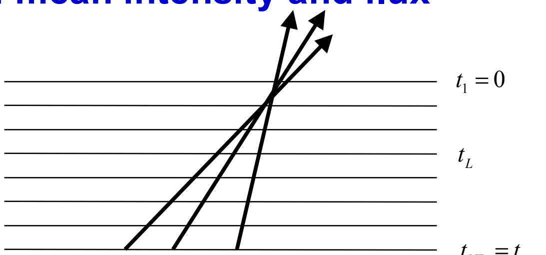
Problem: Scattering $\kappa_e = n_e \sigma_e \quad , \quad \eta_e = \kappa_e J = \kappa_e \frac{1}{2} \int_{-1}^{1} I(\mu) d\mu$

Requires iteration

Determination of mean intensity and flux

Discrete angular points

$\mu_j \ , \ j = 1 \cdots NA$



Solution along each ray

$I_j^{\pm} = \sum_{L=1}^{ND} S_L w_{Lj}$ weights depend on μ_j

Angular integration

$J = \frac{1}{2} \sum_{j=1}^{NA} I_j w_j \quad , \quad \mathrm{H} = \frac{1}{2} \sum_{j=1}^{NA} I_j \mu_j w_j$

Gauss integration with 3 points sufficient for pp-RT

Alternative: numerical integration of moment equation

Problem: numerical approximation of $E_2(\tau)$

Spherical geometry

Impact parameters

$P_j \ , \ j = 1 \cdots NP \ , \ NP = ND + NC$

$P_1 = 0, \cdots, P_{NC}$ intersecting the core

$P_{NC+1} = 1, \cdots, P_{NP} = R_{\max}$

Z_i points

$Z_i = +\sqrt{r_l^2 - P_j^2}, \ i = 1 \cdots ND$ intersecting the core

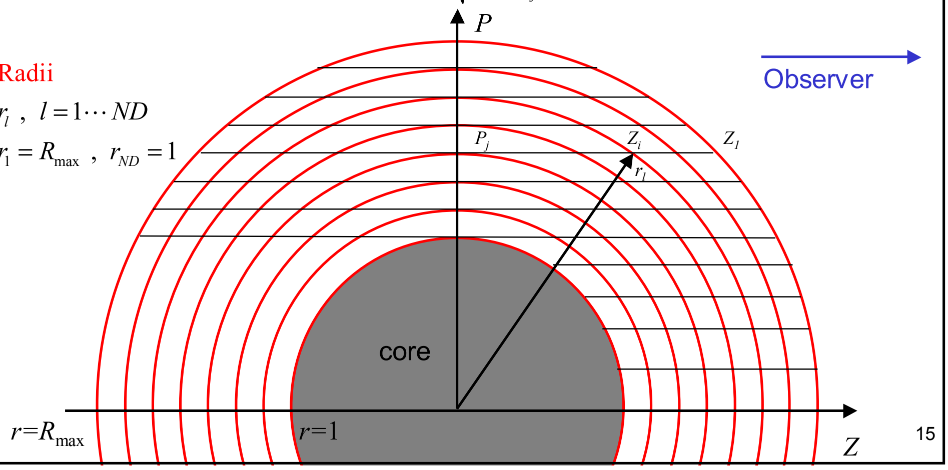
$Z_i = +\sqrt{r_l^2 - P_j^2}, \ l = 1 \cdots NP + 1 - j, i = l$

$Z_i = -\sqrt{r_l^2 - P_j^2}, \ l = 1 \cdots NP + 1 - j, i = 2(NP + 1 - j) - l$

Radii

$r_l \ , \ l = 1 \cdots ND$

$r_1 = R_{\max} \ , \ r_{ND} = 1$



Spherical geometry

Numerical integration on this grid, e.g.:

Optical depth

$$\tau_i = \int_{Z_1}^{Z_i} \kappa\left(r = \sqrt{Z_i^2 + P_j^2}\right) dZ$$

Emergent intensity

$$I_j^+ = \int_{\tau'=0}^{\tau_{\max}} S(\tau') e^{-\tau'} d\tau' = \sum_{i=1}^{i_{\max}} S_l w_i$$

Emergent flux

$$H^+ = \frac{1}{2} \int_{P=0}^{R_{\max}} I^+(P) P dP = \frac{1}{2} \sum_{j=1}^{NP} I_j^+ w_j$$

Solution as boundary-value problem
Feautrier scheme

Radiation transfer equation along a ray:

$$\pm\frac{dI_\nu^\pm(\tau)}{d\tau}=I_\nu^\pm(\tau)-S_\nu(\tau)$$

$$\text{pp: } d\tau=\kappa\frac{dt}{d\mu}$$

$$\text{sp: } d\tau=-\kappa dZ$$

Two differential equations for inbound and outbound rays
Definitions by Feautrier (1964):

$$u=\frac{1}{2}\left(I^+ + I^-\right) \quad \text{symmetric, intensity-like}$$

$$v=\frac{1}{2}\left(I^+ - I^-\right) \quad \text{antisymmetric, flux-like}$$

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Feautrier scheme

Addition and subtraction of both DEQs:

$$\frac{dv(\tau)}{d\tau}=u(\tau)-S_\nu(\tau) \qquad (1)$$

$$\frac{du(\tau)}{d\tau}=v(\tau) \qquad (2)$$

$$\Rightarrow\frac{d^2u(\tau)}{d\tau^2}=u(\tau)-S_\nu(\tau)$$

One DEQ of second order instead of two DEQ of first order

Feautrier scheme

Boundary conditions (pp-case)

Outer boundary

$$I^-(\tau=0)=0 \rightarrow u(\tau=0)=v(\tau=0)$$

$$(2) \quad \Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=0}=u(\tau=0)$$

... with irradiation

$$I^-(\tau=0)=I_0^- \rightarrow u-v=I^-$$

$$\Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=0}=u(\tau=0)-I_0^-$$

Inner boundary

$$I^+(\tau=\tau_{\max})=I^+_{\tau_{\max}} \rightarrow u(\tau_{\max})+v(\tau_{\max})=I^+_{\tau_{\max}}$$

$$(2) \quad \Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=\tau_{\max}}=I^+_{\tau_{\max}}-u(\tau_{\max})$$

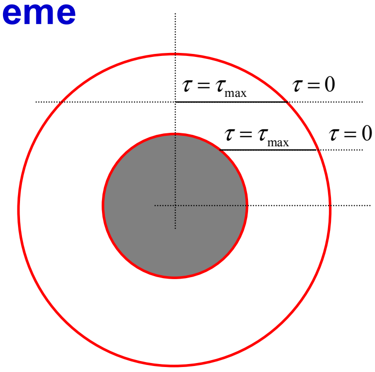
Schuster boundary-value problem

Feautrier scheme

Boundary conditions (spherical)

Core rays:

Like pp-case



Non-core rays:

Restrict to one quadrant (symmetry) inner boundary at Z=0:

$$I^+(\tau=\tau_{\max})=I^-(\tau=\tau_{\max}) \rightarrow v(\tau_{\max})=0$$

$$\Rightarrow \left.\frac{du(\tau)}{d\tau}\right|_{\tau=\tau_{\max}}=0$$

Finite differences

Approximation of the derivatives by finite differences:

$$u-\frac{d^2u}{d\tau^2}=S \quad \text{discretization on a } \tau-\text{scale}$$

first derivative at intermediate points:

$$\tau_{i+1/2}=\frac{1}{2}(\tau_{i+1}+\tau_i)$$

$$\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i+1/2}}\approx\frac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i} \quad \text{and} \quad \left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i-1/2}}\approx\frac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}$$

second derivative:

$$\left.\frac{d}{d\tau}\left(\frac{du(\tau)}{d\tau}\right)\right|_{\tau_i}=\frac{\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i+1/2}}-\left.\frac{du(\tau)}{d\tau}\right|_{\tau_{i-1/2}}}{\tau_{i+1/2}-\tau_{i-1/2}}$$

$$\left.\frac{du^2(\tau)}{d\tau^2}\right|_{\tau_i}\approx\frac{\frac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i}-\frac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}}{\frac{1}{2}(\tau_{i+1}-\tau_{i-1})}$$

Finite differences

Approximation of the derivatives by finite differences:

$$u-\frac{d^2u}{d\tau^2}=S \quad \text{discretisation on a } \tau-\text{scale}$$

$$\Rightarrow u_i-\frac{\frac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i}-\frac{u_i-u_{i-1}}{\tau_i-\tau_{i-1}}}{\frac{1}{2}(\tau_{i+1}-\tau_{i-1})}=S_i \quad , \quad i=2\cdots ND-1$$

$$\boxed{\begin{aligned}&\Rightarrow -A_iu_{i-1}+B_iu_i-C_iu_{i+1}=S_i \quad , \quad i=2\cdots ND-1\\&A_i=\left[\frac{1}{2}(\tau_i-\tau_{i-1})(\tau_{i+1}-\tau_{i-1})\right]^{-1}\\&C_i=\left[\frac{1}{2}(\tau_{i+1}-\tau_i)(\tau_{i+1}-\tau_{i-1})\right]^{-1}\\&B_i=1+A_i+C_i\end{aligned}}$$

Discrete boundary conditions

Outer boundary: first order

$$\left.\frac{du}{d\tau}\right|_{\tau_1} = u_1 \rightarrow \frac{u_2 - u_1}{\tau_2 - \tau_1} = u_1$$

$$\rightarrow u_1 - \frac{u_2 - u_1}{\tau_2 - \tau_1} = 0$$

$$\Rightarrow B_1 u_1 - C_1 u_2 = 0$$

$$C_1 = [\tau_2 - \tau_1]^{-1} \quad , \quad B_1 = 1 + C_1$$

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Discrete boundary conditions

Numerically better is a second-order condition:
Taylor expansion of u(τ) around τ_1:

$$u_2 = u_1 + (\tau_2 - \tau_1)\left.\frac{du}{d\tau}\right|_{\tau_1} + \frac{1}{2}(\tau_2 - \tau_1)^2 \left.\frac{d^2u}{d\tau^2}\right|_{\tau_1}$$

boundary condition DEQ

$$u_2 = u_1 + (\tau_2 - \tau_1) u_1 + \frac{1}{2}(\tau_2 - \tau_1)^2 (u_1 - S_1)$$

$$u_1 + 2\frac{u_1}{\tau_2 - \tau_1} + 2\frac{u_1 - u_2}{(\tau_2 - \tau_1)^2} = S_1 \quad \Rightarrow \quad \boxed{B_1 u_1 - C_1 u_2 = S_1}$$

$$C_1 = 2[\tau_2 - \tau_1]^{-2} \quad , \quad B_1 = 1 + 2[\tau_2 - \tau_1]^{-1} + 2[\tau_2 - \tau_1]^{-2}$$

Discrete boundary conditions

Inner boundary: first order

$$\left.\frac{du(\tau)}{d\tau}\right|_{\tau=\tau_{\max}} = \begin{cases} I^+ - u_{ND} & \text{pp or core rays} \\ 0 & \text{non-core rays} \end{cases} \rightarrow \frac{u_{ND-1}-u_{ND}}{\tau_{ND-1}-\tau_{ND}} = \begin{cases} I^+ - u_{ND} \\ 0 \end{cases}$$

$$\rightarrow \left.\begin{matrix} u_{ND} \\ 0 \end{matrix}\right\} + \frac{u_{ND}-u_{ND-1}}{\tau_{ND}-\tau_{ND-1}} = \begin{cases} I^+ \\ 0 \end{cases}$$

$$\Rightarrow -A_{ND}u_{ND-1} + B_{ND}u_{ND} = S^*_{ND}$$

$$A_{ND} = \left[\tau_{ND}-\tau_{ND-1}\right]^{-1} \quad , \quad B_{ND} = \begin{cases} 1 + A_{ND} \\ A_{ND} \end{cases} \quad , \quad S^*_{ND} = \begin{cases} I^+ \\ 0 \end{cases}$$

Discrete boundary conditions

Outer boundary: second order

$$u_{ND-1} = u_{ND} + \left(\tau_{ND-1}-\tau_{ND}\right)\underbrace{\left.\frac{du}{d\tau}\right|_{\tau_{ND}}}_{\text{boundary condition}} + \frac{1}{2}\left(\tau_{ND-1}-\tau_{ND}\right)^2 \underbrace{\left.\frac{d^2u}{d\tau^2}\right|_{\tau_{ND}}}_{\text{DEQ}}$$

$$u_{ND-1} = u_{ND} - \left(\tau_{ND}-\tau_{ND-1}\right)\left\{\begin{matrix} I^+ - u_{ND} & \text{core} \\ 0 & \text{non-core} \end{matrix}\right\}$$

$$+\frac{1}{2}\left(\tau_{ND}-\tau_{ND-1}\right)^2\left(u_{ND}-S_{ND}\right)$$

$$u_{ND} - 2\frac{I^+ - u_{ND}}{\tau_{ND}-\tau_{ND-1}} + 2\frac{u_{ND}-u_{ND-1}}{\left(\tau_{ND}-\tau_{ND-1}\right)^2} = S^*_{ND} \Rightarrow \boxed{B_{ND}u_{ND} - A_{ND}u_{ND-1} = S^*_{ND}}$$

$$A_{ND} = 2\left[\tau_{ND}-\tau_{ND-1}\right]^{-2} \quad , \quad B_{ND} = 1 + 2\left[\tau_{ND}-\tau_{ND-1}\right]^{-1} + 2\left[\tau_{ND}-\tau_{ND-1}\right]^{-2}$$

$$S^*_{ND} = S_{ND} + 2\left[\tau_{ND}-\tau_{ND-1}\right]^{-1} I^+$$

Linear system of equations

$$
\begin{bmatrix}
B_1 & -C_1 & & & & 0 \\
-A_2 & B_2 & -C_2 & & & \\
& \ddots & \ddots & \ddots & & \\
& & \ddots & \ddots & \ddots & \\
0 & & & \ddots & \ddots & -C_{ND-1} \\
& & & & -A_{ND} & B_{ND}
\end{bmatrix}
\cdot
\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{ND-1} \\ u_{ND} \end{bmatrix}
=
\begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_{ND-1} \\ W_{ND} \end{bmatrix}
=
\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{ND-1} \\ S^*_{ND} \end{bmatrix}
$$

Linear system for u_i

Use Gauss-Jordan elimination for solution

Upper diagonal matrix

1st step:

$$
\begin{bmatrix}
1 & -\tilde{C}_1 & & & 0 \\
& 1 & -\tilde{C}_2 & & \\
& & \ddots & \ddots & \\
& & & \ddots & \ddots \\
0 & & & \ddots & -\tilde{C}_{ND-1} \\
& & & & 1
\end{bmatrix}
\cdot
\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{ND-1} \\ u_{ND} \end{bmatrix}
=
\begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \vdots \\ \tilde{W}_{ND-1} \\ \tilde{W}_{ND} \end{bmatrix}
$$

$$
\boxed{
\begin{array}{lll}
i=1 & \tilde{C}_1 = B_1^{-1} C_1 & \tilde{W}_1 = B_1^{-1} W_1 \\
i=2\cdots ND & \tilde{C}_i = \left(B_i - A_i\tilde{C}_{i-1}\right)^{-1} C_i & \tilde{W}_i = \left(B_i - A_i\tilde{C}_{i-1}\right)^{-1}\left(W_i + A_i\tilde{W}_{i-1}\right)
\end{array}
}
$$

Back-substitution

2nd step:

$$\boxed{\begin{array}{ll} i = ND & u_{ND} = \tilde{W}_{ND} \\ i = ND-1\cdots 1 & u_i = \tilde{W}_i + \tilde{C}_i u_{i+1} \end{array}}$$

Solution fulfils differential equation as well as both boundary conditions

Remark: for later generalization the matrix elements are treated as matrices (non-commutative)

Solution with linear dependent source function

Coherent scattering:

General form (complete redistribution)

$$S_\nu = \alpha J_\nu + \beta \quad , \quad J_\nu = \frac{1}{2}\int_{\mu=-1}^{1} I_\nu(\mu)d\mu = \int_{\mu=0}^{1} u_\nu(\mu)d\mu$$

Thomson scattering

$$\kappa_e = n_e\sigma_e \quad , \quad \eta_e = \kappa_e J$$

$$\Rightarrow S^{tot} = \frac{\eta+\eta_e}{\kappa+\kappa_e} = \frac{\kappa}{\kappa+\kappa_e}\frac{\eta}{\kappa} + \frac{\kappa_e}{\kappa+\kappa_e}\frac{\eta_e}{\kappa_e} = \frac{\kappa+\kappa_e-\kappa_e}{\kappa+\kappa_e}S + \frac{\kappa_e}{\kappa+\kappa_e}J$$

$$S^{tot} = (1-\beta_e)S + \beta_e J \quad , \quad \beta_e = \frac{\kappa_e}{\kappa+\kappa_e}$$

Results in coupling of equations for all directions

$$\boxed{u(\tau,\mu) - \beta_e(\tau)\int_{\mu=0}^{1} u(\tau,\mu)\,d\mu - \frac{d^2u(\tau,\mu)}{d\tau^2} = [1-\beta_e(\tau)]S}$$

Discretization

Generalization of Feautrier scheme to a block-matrix scheme:

Angular discretization: $\int\limits_{\mu=0}^{1} u(\tau,\mu)\,d\mu \approx \sum\limits_{j=1}^{NA} u_{ji} w_j$

Depth discretization as previously:

$$-A_i u_{i-1} + B_i u_i - C_i u_{i+1} = W_i$$

equations for vector u_i

Discretization

Identical lines

pp-case identical in all depths

$$u(\tau,\mu) - \beta_e(\tau) \int\limits_{\mu=0}^{1} u(\tau,\mu)\,d\mu - \frac{d^2 u(\tau,\mu)}{d\tau^2} = [1-\beta_e(\tau)]S$$

$$\rightarrow \quad -A_i u_{i-1} + B_i u_i - C_i u_{i+1} = W_i$$

equations for vector u_i

$$u_i = \begin{bmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_{NA} \end{bmatrix}, \; A_i = \begin{pmatrix} A_i & & & \\ & \ddots & & 0 \\ & & A_i & \\ & 0 & & \ddots \\ & & & & A_i \end{pmatrix}, \; C_i = \begin{pmatrix} C_i & & & \\ & \ddots & & 0 \\ & & C_i & \\ & 0 & & \ddots \\ & & & & C_i \end{pmatrix}$$

$$B_i = \begin{pmatrix} B_i & & & \\ & \ddots & & 0 \\ & & B_i & \\ & 0 & & \ddots \\ & & & & B_i \end{pmatrix} - \beta_e(i) \begin{pmatrix} w_1 & \cdots & \cdots & \cdots & w_{NA} \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ w_1 & \cdots & \cdots & \cdots & w_{NA} \end{pmatrix}, \; W_i = \begin{bmatrix} (1-\beta_e(i))S_i \\ \vdots \\ (1-\beta_e(i))S_i \\ \vdots \\ (1-\beta_e(i))S_i \end{bmatrix}$$

Block matrix

$$
\begin{pmatrix}
\begin{bmatrix} B_{1;1,1} & \cdots & B_{1;1,NA} \\ \vdots & \ddots & \vdots \\ B_{1;NA,1} & \cdots & B_{1;NA,NA} \end{bmatrix} & \begin{bmatrix} -C_1 & & \\ & \ddots & \\ & & -C_1 \end{bmatrix} & & & \\
\begin{bmatrix} -A_2 & & \\ & \ddots & \\ & & -A_2 \end{bmatrix} & \begin{bmatrix} B_{2;1,1} & \cdots & B_{2;1,NA} \\ \vdots & \ddots & \vdots \\ B_{2;NA,1} & \cdots & B_{2;NA,NA} \end{bmatrix} & \begin{bmatrix} -C_2 & & \\ & \ddots & \\ & & -C_2 \end{bmatrix} & & \\
& & \ddots & & \\
& & & \begin{bmatrix} B_{ND-1;1,1} & \cdots & B_{ND-1;1,NA} \\ \vdots & \ddots & \vdots \\ B_{ND-1;NA,1} & \cdots & B_{ND-1;NA,NA} \end{bmatrix} & \begin{bmatrix} -C_{ND-1} & & \\ & \ddots & \\ & & -C_{ND-1} \end{bmatrix} \\
& & & \begin{bmatrix} -A_{ND} & & \\ & \ddots & \\ & & -A_{ND} \end{bmatrix} & \begin{bmatrix} B_{ND;1,1} & \cdots & B_{ND;1,NA} \\ \vdots & \ddots & \vdots \\ B_{ND;NA,1} & \cdots & B_{ND;NA,NA} \end{bmatrix}
\end{pmatrix}
\cdot
\begin{bmatrix}
\begin{bmatrix} u_{1;1} \\ \vdots \\ u_{1;NA} \end{bmatrix} \\
\begin{bmatrix} u_{2;1} \\ \vdots \\ u_{2;NA} \end{bmatrix} \\
\vdots \\
\begin{bmatrix} u_{ND-1;1} \\ \vdots \\ u_{ND-1;NA} \end{bmatrix} \\
\begin{bmatrix} u_{ND;1} \\ \vdots \\ u_{ND;NA} \end{bmatrix}
\end{bmatrix}
=
\begin{bmatrix}
\begin{bmatrix} W_{1;1} \\ \vdots \\ W_{1;NA} \end{bmatrix} \\
\begin{bmatrix} W_{2;1} \\ \vdots \\ W_{2;NA} \end{bmatrix} \\
\vdots \\
\begin{bmatrix} W_{ND-1;1} \\ \vdots \\ W_{ND-1;NA} \end{bmatrix} \\
\begin{bmatrix} W_{ND;1} \\ \vdots \\ W_{ND;NA} \end{bmatrix}
\end{bmatrix}
$$

$$W_{i;j} = \left(1 - \beta_e(i)\right) S_i \quad , \quad i = 1 \cdots ND - 1$$

$$W_{ND;j} = \left(1 - \beta_e(ND)\right) S_{ND} + 2\left(\tau_{ND} - \tau_{ND-1}\right)^{-1} I_j^+$$

Feautrier scheme

Thermal source function

- Decoupled equation for each direction
- Separate Feautrier scheme for each ray
- ND•5 multiplications

Thermal source function + coherent scattering (e.g. Thomson)

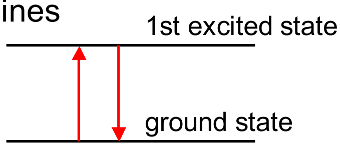
- Coupled equations for all directions
- Feautrier scheme in block matrix form
- ND•NA^3 multiplications (matrix inversions)

Spherical geometry

- Radius dependent angular integration $\mu_{j,l} = \left(1 - \frac{P_j^2}{r_l^2}\right)^{1/2} \ , \ j = 1 \cdots JMAX(l) \ , \ JMAX(l) = NP + 1 - l$
- Block matrix size depends on radius
- ~ND^4 multiplications

Solution with line scattering

Two-level atom, complete redistribution
Photon conservation within a spectral line
Approximately realized for resonance lines



$\kappa(\nu)=\kappa_L\varphi(\nu)$, $\eta(\nu)=\eta_L\varphi(\nu)$

photon conservation:

$$\int_{Line}\eta(\nu)d\nu=\int_{Line}\kappa(\nu)J_\nu(\nu)d\nu$$

$$\eta_L=\kappa_L\int_{Line}J_\nu(\nu)\varphi_\nu d\nu$$

$$\boxed{S_L=\int_{Line}J_\nu(\nu)\varphi_\nu d\nu}$$

(generalized) scattering intergral $S_L=\alpha\bar{J}_L+\beta$

The frequency independent line source function (☛) is a weighted mean of the mean intensity

Solution with line scattering (non-coherent scattering)

Transfer equation:

$$\mu\frac{dI_\nu(\nu,\mu,\tau)}{d\tau(\nu)}=I_\nu(\nu,\mu,\tau)-S_L(\tau)$$

$$\mu\frac{dI_\nu(\nu,\mu,\tau)}{d\tau(\nu)}=I_\nu(\nu,\mu,\tau)-\frac{1}{2}\int_{Line}\varphi(\nu)\int_{\mu=-1}^{+1}I_\nu(\nu,\mu,\tau)d\mu d\tau$$

Each one equation for each angular and frequency point
Each equation contains intensities from all other points
⇒coupling of all transfer equations to an integro-differential-equation
This system is linear with respect to the intensity
⇒solution with Gauss-Jordan elimination

Solution with line scattering

Discretization:

$$u(\nu,\mu,\tau)-\alpha(\tau)\int\limits_{Line}\varphi(\nu)\int\limits_{\mu=-1}^{+1}u(\nu,\mu,\tau)d\mu d\nu-\frac{d^2u(\nu,\mu,\tau)}{d\tau^2(\nu)}=\beta(\tau)$$

Analogous solution as with coherent scattering

Scattering integral:

$$\overline{J}_L=\int\limits_{\nu_0-\Delta\nu}^{\nu_0+\Delta\nu}J(\nu)\varphi(\nu)d\nu\rightarrow\sum_{k=1}^{NF}J_k w_k'$$

With appropriate weights including the line profile function (e.g. Gauss function)

$$\overline{J}_L\rightarrow\sum_{j=1}^{NA}\sum_{k=1}^{NF}u_{jk}w_j w_k'$$

Linear system of equations

$$\begin{bmatrix} B_1 & -C_1 & & & & 0 \\ -A_2 & B_2 & -C_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & & & \ddots & \ddots & -C_{ND-1} \\ & & & & -A_{ND} & B_{ND} \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_{ND-1} \\ u_{ND} \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ \vdots \\ W_{ND-1} \\ W_{ND} \end{bmatrix}$$

Linear system for u_i

Use Gauss-Jordan elimination for solution

Block matrix

$$u_i - \alpha_i \sum_{j=1}^{NA} \sum_{k=1}^{NF} u_{jk} w_j w'_k - \frac{\dfrac{u_{i+1}-u_i}{\tau_{i+1}-\tau_i} - \dfrac{u_i - u_{i-1}}{\tau_i - \tau_{i-1}}}{\frac{1}{2}\left(\tau_{i+1}-\tau_{i-1}\right)} = \beta_i$$

no longer identical, depend on k but not on j

$$u_i = \begin{bmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_{NA} \end{bmatrix} \quad u_j = \begin{bmatrix} u_{j,1} \\ \vdots \\ u_{j,k} \\ \vdots \\ u_{j,NF} \end{bmatrix}, \quad A_i = \begin{pmatrix} A_{ijk} & & & \\ & \ddots & 0 & \\ & & A_{ijk} & \\ & 0 & & \ddots & \\ & & & & A_{ijk} \end{pmatrix}, \quad C_i = \begin{pmatrix} C_{ijk} & & & \\ & \ddots & 0 & \\ & & C_{ijk} & \\ & 0 & & \ddots & \\ & & & & C_{ijk} \end{pmatrix}$$

$$B_i = \begin{pmatrix} B_{ijk} & & & \\ & \ddots & 0 & \\ & & B_{ijk} & \\ & 0 & & \ddots & \\ & & & & B_{ijk} \end{pmatrix} - \alpha(i) \left(\boxed{\begin{matrix} w_1 w'_1 & \cdots & w_1 w'_{NF} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ w_1 w'_1 & \cdots & w_1 w'_{NF} \end{matrix}} \cdots \boxed{\begin{matrix} w_{NA} w'_1 & \cdots & w_{NA} w'_{NF} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ w_{NA} w'_1 & \cdots & w_{NA} w'_{NF} \end{matrix}} \right), \quad W_i = \begin{bmatrix} \beta(i) \\ \vdots \\ \beta(i) \\ \vdots \\ \beta(i) \end{bmatrix}$$

$$B_{ijk} = 1 + A_{jk} + C_{ijk}$$

Identical lines

Solution with line scattering

The modified Feautrier scheme is not very efficient

pp: inversion of matrices of rank NA•NF ~ 3 •5

ND•NA^3 •NF^3 multiplications

sp: inversion of matrices of rank NA•NF ~ 70 •5

ND•ND^3 •NF^3 multiplications

Repeated calculation of identical scattering integrals at each frequency point

Rybicki scheme

Rybicki (1971)

Coherent scattering, e.g. Thomson scattering $S_\nu^{tot} = (1-\beta_e)S_\nu + \beta_e J_\nu$

Again: Feautrier transfer equation:

$$u(\tau,\mu) - \beta_e(\tau)J(\tau) - \frac{d^2u(\tau,\mu)}{d\tau^2} = [1-\beta_e(\tau)]S(\tau)$$

Discretization:

$$-A_{i,j}u_{i-1,j} + B_{i,j}u_{i,j} - C_{i,j}u_{i+1,j} - \beta_i^e J_i = \left[1-\beta_i^e\right]S_i$$

$$j = 1\cdots NA \text{ (pp)} \quad , \quad j = 1\cdots JMAX \text{ (sp)}$$

$$i = (1), 2\cdots ND-1, (ND)$$

A_i, B_i, C_i like in usual Feautrier scheme

But now: additional variable J_i

With additional equations $J_i = \sum_j u_{i,j} w_{i,j}$

Use alternative grouping

Rybicki scheme

$$\left(\begin{array}{ccccc|ccc}
T_1 & & & & & & U_1 & \\
 & \ddots & & 0 & & & \vdots & \\
 & & T_j & & & & U_j & \\
 & 0 & & \ddots & & & \vdots & \\
 & & & & T_{NA} & & U_{NA} & \\ \hline
 & & & & & -1 & & 0 \\
W_1 & \cdots & W_j & \cdots & W_{NA} & & \ddots & \\
 & & & & & 0 & & -1
\end{array}\right)
\begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_j \\ \vdots \\ \vec{u}_{NA} \\ J_1 \\ \vdots \\ J_{ND} \end{bmatrix}
=
\begin{bmatrix} \vec{K}_1 \\ \vdots \\ \vec{K}_j \\ \vdots \\ \vec{K}_{NA} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{u}_j = \begin{bmatrix} u_{1,j} \\ \vdots \\ u_{i,j} \\ \vdots \\ u_{ND,j} \end{bmatrix} \qquad \vec{K}_j = \begin{bmatrix} \left(1-\beta_1^e\right)S_1 \\ \vdots \\ \left(1-\beta_i^e\right)S_i \\ \vdots \\ \left(1-\beta_{ND}^e\right)S_{ND} \end{bmatrix}$$

$$W_j = \begin{pmatrix} w_{i,j} & & & \\ & \ddots & 0 & \\ & & w_{i,j} & \\ & 0 & \ddots & \\ & & & w_{ND,j} \end{pmatrix}$$

$$T_j = \begin{pmatrix} B_{1,j} & -C_{1,j} & & & \\ & \ddots & & 0 & \\ & -A_{i,j} & B_{i,j} & -C_{i,j} & \\ & 0 & & \ddots & \\ & & & -A_{ND,j} & B_{ND,j} \end{pmatrix} \qquad U_j = \begin{pmatrix} -\beta_1^e & & & \\ & \ddots & 0 & \\ & & -\beta_i^e & \\ & 0 & \ddots & \\ & & & -\beta_{ND}^e \end{pmatrix}$$

$T_j\vec{u}_j$ represents $u - \frac{d^2u}{d\tau^2}$, $U_j\vec{J}$ describes $-\beta_e J$ for the ray $j \rightarrow T_j\vec{u}_j + U_j\vec{J} = \vec{K}_j$

Scattering integral is represented by last block line $\sum_{j=1}^{NA} W_j\vec{u}_j - \vec{J} = 0$

Rybicki scheme

1st step: diagonal elements to unit matrixes

$$
\left(\begin{array}{ccccc|ccc}
1 & & & & & & \tilde{U}_1 & \\
 & \ddots & & 0 & & & \vdots & \\
 & & 1 & & & & \tilde{U}_j & \\
 & 0 & & \ddots & & & \vdots & \\
 & & & & 1 & & \tilde{U}_{NA} & \\
\hline
 & & & & & -1 & & 0 \\
W_1 & \cdots & W_j & \cdots & W_{NA} & & \ddots & \\
 & & & & & 0 & & -1
\end{array}\right)
\begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_j \\ \vdots \\ \vec{u}_{NA} \\ J_1 \\ \vdots \\ J_{ND} \end{bmatrix}
=
\begin{bmatrix} \vec{\tilde{K}}_1 \\ \vdots \\ \vec{\tilde{K}}_j \\ \vdots \\ \vec{\tilde{K}}_{NA} \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

$$\tilde{U}_j = T_j^{-1} U_j \ , \ \vec{\tilde{K}}_j = T_j^{-1} \vec{K}_j$$

Corresponds to the solution of NA tri-diagonal systems of equations of rank ND → cpu-time ~ NA•ND²

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Rybicki scheme

2nd step: W_j to zero

$$
\begin{pmatrix}
1 & & & & & \tilde{U}_1 \\
 & \ddots & & 0 & & \vdots \\
 & & 1 & & & \tilde{U}_j \\
 & 0 & & \ddots & & \vdots \\
 & & & & 1 & \tilde{U}_{NA} \\
0 & \cdots & 0 & \cdots & 0 & W
\end{pmatrix}
\begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_j \\ \vdots \\ \vec{u}_{NA} \\ \vec{J} \end{bmatrix}
=
\begin{bmatrix} \vec{\tilde{K}}_1 \\ \vdots \\ \vec{\tilde{K}}_j \\ \vdots \\ \vec{\tilde{K}}_{NA} \\ \vec{Q} \end{bmatrix}
$$

$$W = -1 - \sum_{j=1}^{NA} W_j \tilde{U}_j \ , \ \vec{Q} = -\sum_{j=1}^{NA} W_j \vec{\tilde{K}}_j$$

Corresponds to cpu-time ~ NA•ND²

Rybicki scheme

3rd step: solve for J

$$W\vec{J} = \vec{Q} \rightarrow \vec{J} = W^{-1}\vec{Q}$$

Corresponds to cpu-time ~ ND^3
Finished if only J is required

4th step: solve for Feautrier variables u:

$$\vec{u}_j = T_j^{-1}\vec{K}_j - T_j^{-1}\vec{U}_j\vec{J}$$

Comparison Rybicki vs. Feautrier

Thomson scattering

	Feautrier	Rybicki
Plane-parallel	C NA^3 ND	C_1 NA ND^2+ $C_2 ND^3$
Spherical	C ND^4	C_1 NP ND^2+ $C_2 ND^3$

Few angular points: take Feautrier
Many angular points: take Rybicki

Solution with line scattering (non-coherent scattering)

Two-level atom or Compton scattering $S_L = \alpha \bar{J}_L + \beta$

Each one block line $T_{jk}\vec{u}_{jk} + U_{jk}\vec{J} = \vec{K}_{jk}$

describes transfer equation for each ray j,k

(direction and frequency dependent!)

$\rightarrow$ Huge system of equations

$$
\begin{pmatrix}
T_1 & & & & & & U_1 & \\
 & \ddots & & 0 & & & \vdots & \\
 & & T_{jk} & & & & U_{jk} & \\
 & 0 & & \ddots & & & \vdots & \\
 & & & & T_{NANF} & & U_{NANF} & \\
 & & & & & -1 & & 0 \\
W_1 & \cdots & W_{jk} & \cdots & W_{NANF} & & \ddots & \\
 & & & & & 0 & & -1
\end{pmatrix}
\begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_{jk} \\ \vdots \\ \vec{u}_{NANF} \\ J_1 \\ \vdots \\ J_{ND} \end{bmatrix}
=
\begin{bmatrix} \vec{K}_1 \\ \vdots \\ \vec{K}_{jk} \\ \vdots \\ \vec{K}_{NANF} \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

Solution with line scattering (non-coherent scattering)

$$
\vec{u}_{jk} = \begin{bmatrix} u_{1,j} \\ \vdots \\ u_{ij} \\ \vdots \\ u_{ND,j} \end{bmatrix}
\qquad
\vec{K}_{jk} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_i \\ \vdots \\ \beta_{ND} \end{bmatrix}
$$

$$
\begin{pmatrix}
T_1 & & & & & & U_1 & \\
 & \ddots & & 0 & & & \vdots & \\
 & & T_{jk} & & & & U_{jk} & \\
 & 0 & & \ddots & & & \vdots & \\
 & & & & T_{NANF} & & U_{NANF} & \\
 & & & & & -1 & & 0 \\
W_1 & \cdots & W_{jk} & \cdots & W_{NANF} & & \ddots & \\
 & & & & & 0 & & -1
\end{pmatrix}
\begin{bmatrix} \vec{u}_1 \\ \vdots \\ \vec{u}_{jk} \\ \vdots \\ \vec{u}_{NANF} \\ J_1 \\ \vdots \\ J_{ND} \end{bmatrix}
=
\begin{bmatrix} \vec{K}_1 \\ \vdots \\ \vec{K}_{jk} \\ \vdots \\ \vec{K}_{NANF} \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

$$
U_{jk} = \begin{pmatrix}
-\alpha_1 & & \\
 & \ddots & 0 \\
 & -\alpha_i & \\
0 & & \ddots \\
 & & -\alpha_{ND}
\end{pmatrix}
$$

$$
T_{jk} = \begin{pmatrix}
B_{1,jk} & -C_{1,jk} & & & \\
 & \ddots & & 0 & \\
 & -A_{i,jk} & B_{i,jk} & -C_{i,jk} & \\
 & 0 & & \ddots & \\
 & & & -A_{ND,jk} & B_{ND,jk}
\end{pmatrix}
$$

$$
W_{jk} = \begin{pmatrix}
w_{i,j}w'_k & & \\
 & \ddots & 0 \\
 & w_{i,j}w'_k & \\
0 & & \ddots \\
 & & w_{ND,j}w'_k
\end{pmatrix}
$$

$$
J_i = \int J_\nu(\tau_i)\varphi(\nu)d\nu
$$

Comparison Rybicki vs. Feautrier

Line scattering or non-coherent scattering, e.g. Compton scattering

	Feautrier	Rybicki
Plane-parallel	C $NA^3 NF^3$ ND	C_1 NA NF ND^2+ $C_2 ND^3$
Spherical	C $NF^3 ND^4$	C_1 NP NF ND^2+ $C_2 ND^3$

Few frequency points: take Feautrier or Rybicki
Many frequency points: take Rybicki
Spherical symmetry: take Rybicki

Variable Eddington factors

0-th moment $J = \frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)d\mu = \int_{\mu=0}^{1} u(\mu)d\mu$

1st moment $H = \frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)\mu d\mu = \int_{\mu=0}^{1} v(\mu)\mu d\mu$

2nd moment $K = \frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)\mu^2 d\mu = \int_{\mu=0}^{1} u(\mu)\mu^2 d\mu$

Eddington factor: $f = K/J$

0-th moment of RT (pp) $\frac{dH_\nu(\nu,\tau)}{d\tau(\nu)} = J_\nu(\nu,\tau) - S_\nu(\nu,\tau)$

1st moment of RT (pp) $\frac{dK_\nu(\nu,\tau)}{d\tau(\nu)} = H_\nu(\nu,\tau)$

Variable Eddington factors

Plane-parallel $\dfrac{d^2 K_\nu(\nu,\tau)}{d\tau^2(\nu)} = J_\nu(\nu,\tau) - S_\nu(\nu,\tau)$

With variable Eddington factor $\dfrac{d^2\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau^2(\nu)} = J_\nu(\nu,\tau) - S_\nu(\nu,\tau)$

With given f and S 2nd-order DEQ for J

Outer boundary: $h(\tau=0) = H(\tau=0)/J(\tau=0)$

$$I^-(\tau=0) \rightarrow u(\tau=0) = v(\tau=0)$$

$$\rightarrow h(\tau=0) = \int_{\mu=0}^{1} u(\mu)\mu d\mu \Big/ \int_{\mu=0}^{1} u(\mu) d\mu$$

$$\left.\frac{d\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau(\nu)}\right|_{\tau=0} = h(\tau=0)J_\nu(\nu,\tau=0)$$

Variable Eddington factors

Inner boundary:

$$h(\tau=\tau_{\max}) = H(\tau=\tau_{\max})/J(\tau=\tau_{\max})$$

$$h(\tau=\tau_{\max}) = \underbrace{\int_{\mu=0}^{1} u(\mu)\mu d\mu}_{\neq H!} \Big/ \underbrace{\int_{\mu=0}^{1} u(\mu) d\mu}_{=J}$$

$$\tau=\tau_{\max}: \quad H = \int_{\mu=0}^{1} v(\mu)\mu d\mu = \int_{\mu=0}^{1} I^+(\mu)\mu d\mu - \int_{\mu=0}^{1} u(\mu)\mu d\mu = H^+ - hJ$$

$$\left.\frac{d\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau(\nu)}\right|_{\tau=\tau_{\max}} = H_\nu^+(\tau=\tau_{\max}) - h(\tau=\tau_{\max})J_\nu(\nu,\tau=\tau_{\max})$$

2nd-order boundary conditions from Taylor series ☚

Solution

Discretization $\rightarrow$ algebraic equation

$$J_\nu(\nu,\tau) - \frac{d^2\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau^2(\nu)} = S_\nu(\nu,\tau)$$

$$\rightarrow -A_i J_{i-1} + B_i J_i - C_i J_{i+1} = S_i$$

Tri-diagonal system, solution analogous to Feautrier scheme ☚

Thomson scattering:

$$S^{tot} = (1-\beta_e)S + \beta_e J$$

$$\rightarrow (1-\beta_e)J_\nu(\nu,\tau) - \frac{d^2\left[f(\nu,\tau)J_\nu(\nu,\tau)\right]}{d\tau^2(\nu)} = (1-\beta_e)S_\nu(\nu,\tau)$$

$$\rightarrow -A_i J_{i-1} + (B_i - \beta_i^e)J_i - C_i J_{i+1} = (1-\beta_i^e)S_i$$

Possible without extra costs ☺

Variable Eddington factors

But the Eddington factors are unknown $\rightarrow$ iteration

1. Formal solution u with $S=B$
2. Start value $f_i \equiv 1/3$
3. Solution of the moment equation for $J \rightarrow S^{\,tot}$
4. Formal solution u with given S $\rightarrow K$
5. New Eddington factors f

$$f_i = \sum_{j=1}^{NA} u_{i,j} w_j \bigg/ \sum_{j=1}^{NA} u_{i,j} w'_j$$

$$w_j \text{ contains } \mu^2 \quad , \quad w'_j \text{ contains } \mu$$

6. Converged?

Variable Eddington factors

0-th moment $J = \frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)d\mu = \int_{\mu=0}^{1} u(\mu)d\mu \;,\; \tilde{J} = r^2 J$

1st moment $H = \frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)\mu d\mu = \int_{\mu=0}^{1} v(\mu)\mu d\mu \;,\; \tilde{H} = r^2 H$

2nd moment $K = \frac{1}{2}\int_{\mu=-1}^{+1} I(\mu)\mu^2 d\mu = \int_{\mu=0}^{1} u(\mu)\mu^2 d\mu \;,\; \tilde{K} = r^2 K$

0-th moment of RT (sp)

$$\frac{d\tilde{H}_\nu(\nu,r)}{dr} = \kappa(\nu,r)\left(\underbrace{\tilde{S}_\nu(\nu,r)}_{=r^2 S} - \tilde{J}_\nu(\nu,r)\right)$$

1st moment of RT (sp)

$$\frac{dK_\nu(\nu,r)}{dr} + \frac{1}{r}\left[3K_\nu(\nu,r) - J_\nu(\nu,r)\right] = -\kappa(\nu,r)H_\nu(\nu,r)$$

Variable Eddington factors

1st moment of RT (sp)

$$\frac{dK_\nu(\nu,r)}{dr} + \frac{1}{r}\left[3K_\nu(\nu,r) - J_\nu(\nu,r)\right] = -\kappa(\nu,r)H_\nu(\nu,r)$$

Eddington factor

$$\frac{d(fJ)}{dr} + \frac{J}{r}[3f - 1] = -\kappa H$$

Introduction of the sphericality factor (Auer 1971),

$$r^2 q(r) = \exp\left(\int_1^r \frac{3f(r')-1}{r' f(r')}dr'\right) \quad , \quad \frac{d\left(r^2 q(r)\right)}{dr} = r^2 q(r)\frac{3f(r)-1}{rf(r)}$$

which corresponds to the integrating factor for the DEQ

$$y' + f(x)y = g(x) \;,\; M(x) = \exp\left(\int f(x)dx\right)$$

Variable Eddington factors

$$\frac{d\left(r^2 q(r) K\right)}{dr} = \frac{d\left(r^2 q(r)\right)}{dr} K + r^2 q(r) \frac{dK}{dr}$$

$$= r^2 q(r) \left[\frac{3f-1}{rf} fJ + \frac{d(fJ)}{dr} \right] = r^2 q(r) \left(-\kappa H\right)$$

1st moment equation:

$$\frac{d\left(r^2 q(r) K\right)}{r^2 q(r) dr} = -\kappa H \rightarrow \frac{d\left(qf\tilde{J}\right)}{q\kappa dr} = -\tilde{H}$$

$$\xrightarrow[dx=-q\kappa dr]{}$$

$$\left.\begin{aligned} \frac{d\tilde{H}}{dx} &= \frac{1}{q}\left(\tilde{J}-\tilde{S}\right) \\ \frac{d\left(qf\tilde{J}\right)}{dx} &= \tilde{H} \end{aligned}\right\} \frac{d^2\left(qf\tilde{J}\right)}{dx^2} = \frac{1}{q}\left(\tilde{J}-\tilde{S}\right)$$

Solution

Iteration of Eddington factors like in pp case

Additional integration of sphericality factor

$$q_i = r_i^{-2} \exp\left(\sum_{l=i}^{ND} \frac{3f_l - 1}{f_l} w_l \right) \;,\; w_l \text{ include weights for } \frac{dr}{r}$$

f=1/3 is a bad starting point, f→1 for r →∞

Computational demand much smaller than for formal solution

Non-coherent scattering and moment equation

Two-level atom or Compton scattering $S_L = \alpha \bar{J} + \beta$

For each frequency point one moment equation of 2nd order for mean intensity and Eddington factor $J_\nu(\nu_k)$, $f_{ik}(\tau_i, \nu_k)$

Coupled by frequency integral $\bar{J} = \int J_\nu \varphi(\nu) d\nu \rightarrow \bar{J} = \sum_{k=1}^{NF} J_k w_k$

pp $$J_\nu(\tau,\nu) - \frac{d^2\left(f(\tau,\nu)J_\nu(\tau,\nu)\right)}{d\tau^2(\nu)} - \alpha \int_\nu J_\nu \varphi(\nu) d\nu = \beta(\tau)$$

sp $$\tilde{J}_\nu(r,\nu) - q(r,\nu)\frac{d^2\left(q(r,\nu)f(r,\nu)\tilde{J}_\nu(r,\nu)\right)}{dx^2(\nu)} - \alpha \int_\nu \tilde{J}_\nu \varphi(\nu) d\nu$$
$$= \tilde{\beta}(r)$$

Stellar Atmospheres: Radiation Transport as Boundary-Value Problem

Non-coherent scattering and moment equation

Feautrier: $$\vec{J}_i = [J_{i1}, \cdots, J_{ik}, \cdots, J_{iNF}]^T \quad , \; i = 1 \cdots ND$$

cpu-time~ND•NF³

$$-A_i \vec{J}_{i-1} + B_i \vec{J}_i - C_i \vec{J}_{i+1} = \vec{\beta}_i$$

$$\frac{d^2(fJ)}{d\tau^2} \qquad \bar{J}_i = \sum_{k=1}^{NF} J_{ik} w_k$$

Rybicki: $$\vec{J}_k = [J_{1k}, \cdots, J_{ik}, \cdots, J_{NDk}]^T \quad , \; k = 1 \cdots NF$$

$$\bar{J} = \left[\bar{J}_1, \cdots, \bar{J}_i, \cdots, \bar{J}_{ND}\right]^T$$

$$T_k \vec{J}_k + U_k \vec{\bar{J}} = K_k \quad , \quad \sum_{k=1}^{NF} W_k \vec{J}_k - \vec{\bar{J}} = 0$$

cpu-time~NF•ND²+ND³

Multi-level atom

Atmospheric structure assumed to be given
(or accounted for by iteration)

Two sets of equations:

Radiative transfer equation for mean intensity

$$J_\nu(\tau,\nu) - \frac{d^2\left(f(\tau,\nu)J_\nu(\tau,\nu)\right)}{d\tau^2(\nu)} = S_\nu(\tau,\nu)$$

$$S_\nu(\tau,\nu) = \sum_{lu} \eta_\nu^{lu} \Big/ \sum_{lu} \kappa^{lu}(\nu) \quad \text{sum over all bb-, bf-, and ff-transitions}$$

Statistical equilibrium

$$P(J_\nu)\vec{n} = \vec{b} \;\;,\;\; \vec{n} = \left[n_1 \cdots n_{NL}\right]^T$$

Both equations are coupled via radiative rates in matrix P

Multi-level atom

Coupling of J twofold:

- Over frequency via SE
- Over depth via RT

$\rightarrow$ Simultaneous solution

$\rightarrow$ non-linear in J

Lambda Iteration:

0. start approximation for n
1. formal solution with given S
2. solution of statistical equilibrium with given J
3. converged?

Not convergent for large optical depths

Complete Linearization

Auer & Mihalas 1969

Newton-Raphson method in $\mathbb{R}^n$

Solution according to Feautrier scheme

Unknown variables:

$$\vec{\psi}_i = \begin{bmatrix} \vec{J}_i \\ \vec{n}_i \end{bmatrix} \ , \ i = 1 \cdots ND \qquad \psi = [\vec{\psi}_1, \cdots, \vec{\psi}_i, \cdots, \vec{\psi}_{ND}]^T$$

Equations:

$$-A_{i,k} J_{i-1,k} + B_{i,k} J_{i,k} - C_{i,k} J_{i+1,k} - S_{i,k}(\vec{n}_i) = 0 \qquad \text{NF transfer equations}$$

$$P(\vec{J}_i)\vec{n}_i - \vec{b}_i = 0 \qquad \text{NL equations for SE}$$

System of the form:

$$f_{i,\alpha}(\psi) = 0 \ , \ \alpha = 1 \cdots NF + NL$$

Complete Linearization

Start approximation: $f_{i,\alpha}(\psi^0) \neq 0$

Now looking for a correction so that

$$f_{i,\alpha}(\psi^0 + \delta\psi) = 0 \ \ \forall \ i, \alpha$$

Taylor series:

$$0 = f_{i,\alpha}(\psi) = f_{i,\alpha}(\psi^0 + \delta\psi)$$

$$= f_{i,\alpha}(\psi^0) + \sum_{i=1}^{ND} \left\{ \sum_{k=1}^{NF} \frac{\partial f_{i,\alpha}}{\partial J_{i,k}} \delta J_{i,k} + \sum_{l=1}^{NL} \frac{\partial f_{i,\alpha}}{\partial n_{i,l}} \delta n_{i,l} \right\} \Bigg|_{\psi^0} + \cdots$$

Linear system of equations for ND(NF+NL) unknowns $\delta J_{i,k} \ , \ \delta n_{i,l}$

Converges towards correct solution

Many coefficients vanish

Complete Linearization - structure

Only neighbouring depth points (2nd order transfer equation with tri-diagonal depth structure and diagonal statistical equations): $f_{i,\alpha}(\psi) = f_{i,\alpha}(\vec{\psi}_{i-1}, \vec{\psi}_i, \vec{\psi}_{i+1})$

Results in tri-diagonal block scheme (like Feautrier)

$$-A_i \delta\vec{\psi}_{i-1} + B_i \delta\vec{\psi}_i - C_i \delta\vec{\psi}_{i+1} = \vec{L}_i$$

$$-\left(\begin{array}{ccc|c} \ddots & 0 & & \\ & A_{i,k} & & 0 \\ 0 & & \ddots & \\ \hline & & & \\ & 0 & & 0 \\ & & & \end{array}\right)\left[\begin{array}{c} \delta\vec{J}_{i-1} \\ \hline \delta\vec{n}_{i-1} \end{array}\right] + \left(\begin{array}{ccc|ccc} \ddots & 0 & & \square & \square & \square \\ & B_{i,k} & & \square & \square & \square \\ 0 & & \ddots & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square \end{array}\right)\left[\begin{array}{c} \delta\vec{J}_{i} \\ \hline \delta\vec{n}_{i} \end{array}\right]$$

$$-\left(\begin{array}{ccc|c} \ddots & 0 & & \\ & C_{i,k} & & 0 \\ 0 & & \ddots & \\ \hline & & & \\ & 0 & & 0 \\ & & & \end{array}\right)\left[\begin{array}{c} \delta\vec{J}_{i+1} \\ \hline \delta\vec{n}_{i+1} \end{array}\right] = -\left[\begin{array}{c} \square \\ \square \\ f_{i,\alpha}(\psi^0) \\ \square \\ \square \\ \square \end{array}\right]$$

Complete Linearization - structure

Transfer equations: coupling of $J_{i-1,k}$, $J_{i,k}$, and $J_{i+1,k}$ at the same frequency point:

$\rightarrow$ Upper left quadrants of A_i, B_i, C_i describe 2nd derivative $\frac{d^2 J}{d\tau^2}$

Source function is local:

$\rightarrow$ Upper right quadrants of A_i, C_i vanish

Statistical equations are local

$\rightarrow$ Lower right and lower left quadrants of A_i, C_i vanish

Complete Linearization - structure

Matrix B_i:

$$\begin{array}{rc} & \begin{array}{cccccc} 1 & \dots & NF & 1 & \dots & NL \end{array} \\ B_i = & \left(\begin{array}{ccc|ccc} \ddots & & 0 & & \vdots & \\ & B_{i,k} & & \cdots & -\dfrac{\partial S_{i,k}}{\partial n_{i,l'}} & \cdots \\ 0 & & \ddots & & \vdots & \\ \hline & \vdots & & & \vdots & \\ \cdots & \sum\limits_{m=1}^{NL} \dfrac{\partial \left(P_i\right)_{l,m}}{\partial J_{i,k'}} n_{i,m} & \cdots & \cdots & \left(P_i\right)_{l,l'} & \cdots \\ & \vdots & & & \vdots & \end{array}\right) \end{array} \begin{array}{c} 1 \\ \vdots \\ NF \\ 1 \\ \vdots \\ NL \end{array}$$

Complete Linearization

Alternative (recommended by Mihalas): solve SE first and linearize afterwards: $P(\vec{J}_i)\vec{n}_i - \vec{b}_i = 0 \;\rightarrow \vec{n}_i = P(\vec{J}_i)^{-1}\vec{b}_i$

Newton-Raphson method:

- Converges towards correct solution
- Limited convergence radius
- In principle quadratic convergence, however, not achieved because variable Eddington factors and τ-scale are fixed during iteration step
- CPU~ND $(NF+NL)^3 \rightarrow$ simple model atoms only
  - Rybicki scheme is no relief since statistical equilibrium not as simple as scattering integral

Temperature Correction Schemes

Motivation

Up to now: Radiation transfer in a given atmospheric structure

No coupling between radiation field and temperature included

$\rightarrow$ Including radiative equilibrium into solution of radiative transfer $\rightarrow$ Complete Linearization for model atmospheres (next chapter)

$\rightarrow$ Separate solution via temperature correction

- \+ Quite simple implementation
- \+ Application within an iteration scheme allows completely linear system $\rightarrow$ next chapter
- – No direct coupling
- – Moderate convergence properties

Temperature correction – basic scheme

0. start approximation for $T(\tau) \leftarrow T_0(\tau)$
1. formal solution $J_\nu = \Lambda_\nu S_\nu(T)$
2. correction $T(\tau) \leftarrow T(\tau) + \Delta T(\tau)$
3. convergence?

Several possibilities for step 2 based on radiative equilibrium or flux conservation

Generalization to non-LTE not straightforward

With additional equations towards full model atmospheres:

- Hydrostatic equilibrium
- Statistical equilibrium

LTE

Strict LTE $S_\nu(\tau) = B_\nu(T(\tau))$

Scattering $S_\nu(\tau) = (1-\beta_e)B_\nu(T(\tau)) + \beta_e J_\nu(\tau)$

Simple correction from radiative equilibrium:

$$\int_{\nu=0}^{\infty} \kappa(\tau,\nu)\left(J_\nu(\tau,\nu) - B_\nu(T(\tau),\nu)\right) d\nu \neq 0$$

$$\xrightarrow[\Delta T]{} \int_{\nu=0}^{\infty} \kappa(\tau,\nu)\left(J_\nu(\tau,\nu) - B_\nu\left[T(\tau) + \Delta T(\tau)\right]\right) d\nu = 0$$

$$\Rightarrow \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu - \Delta T \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)}\right) d\nu = 0$$

$$\Rightarrow \Delta T = \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu\right) d\nu \Big/ \int_{\nu=0}^{\infty} \kappa \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)} d\nu = \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu\right) d\nu \Big/ \int_{\nu=0}^{\infty} \kappa \Delta B_\nu d\nu$$

LTE

Strict LTE $S_\nu(\tau) = B_\nu(T(\tau))$

Scattering $S_\nu(\tau) = (1-\beta_e)B_\nu(T(\tau)) + \beta_e J_\nu(\tau)$

Simple correction from radiative equilibrium:

$$\int_{\nu=0}^{\infty} \kappa(\tau,\nu)\left(J_\nu(\tau,\nu) - B_\nu(T(\tau),\nu)\right) d\nu \neq 0$$

$$\xrightarrow[\Delta T]{} \int_{\nu=0}^{\infty} \kappa(\tau,\nu)\left(J_\nu(\tau,\nu) - B_\nu\left[T(\tau) + \Delta T(\tau)\right]\right) d\nu = 0$$

$$\Rightarrow \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu - \Delta T \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)}\right) d\nu = 0$$

$$\Rightarrow \Delta T = \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu\right) d\nu \Big/ \int_{\nu=0}^{\infty} \kappa \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)} d\nu$$

LTE

Problem:

$$\Delta T = \int_{\nu=0}^{\infty} \kappa\left(J_\nu - B_\nu\right) d\nu \Big/ \int_{\nu=0}^{\infty} \kappa \left.\frac{\partial B_\nu}{\partial T}\right|_{T=T(\tau)} d\nu$$

$$J_\nu \xrightarrow[\tau\to\infty]{} B_\nu \quad \text{independent of the temperature} \;\Rightarrow \Delta T \to 0$$

Gray opacity (κ independent of frequency):

$$\int_{\nu=0}^{\infty} \kappa(\nu)\left(J_\nu - B_\nu\right) d\nu \to \kappa(J - B)$$

$$\to \kappa(J - B - \Delta B) = 0$$

$$\to \kappa(J - B) = \kappa\Delta B$$

$$\xrightarrow[\text{0.Moment equation}]{} \frac{dH}{dt} = \kappa\Delta B$$

deviation from constant flux provides temperature correction

Unsöld-Lucy correction

Unsöld (1955) for gray LTE atmospheres, generalized by Lucy (1964) for non-gray LTE atmospheres

0-th moment: $\frac{dH_\nu}{dt} = \kappa_\nu \left(J_\nu - B_\nu\right)$

$\int \cdots d\nu \rightarrow \frac{dH}{d\tau} = \frac{\kappa_J}{\kappa_B} J - B \ , \ \kappa_B B = \int_{\nu=0}^{\infty} \kappa_\nu B_\nu d\nu \ , \ \kappa_J J = \int_{\nu=0}^{\infty} \kappa_\nu J_\nu d\nu \ , \ d\tau = \kappa_B dt$

1st moment: $\frac{dK_\nu}{dt} = \kappa_\nu H_\nu$

$\int \cdots d\nu \rightarrow \frac{dK}{d\tau} = \frac{\kappa_H}{\kappa_B} H \ , \ \kappa_H H = \int_{\nu=0}^{\infty} \kappa_\nu H_\nu d\nu$

now new quantities J', H', K' fulfilling radiative equilibrium (local) and flux conservation (non local)

radiative equilibrium: $\frac{dH'}{d\tau} = \frac{\kappa_J}{\kappa_B} J' - B' = 0$

flux conservation: $\frac{dK'}{d\tau} = \frac{\kappa_H}{\kappa_B} H' = \frac{\kappa_H}{\kappa_B} \frac{\sigma}{4\pi} T_{\text{eff}}^4$

Unsöld-Lucy correction

Now corrections to obtain new quantities:

$\Delta X = X' - X$

$\frac{d\Delta K}{d\tau} = \frac{\kappa_H}{\kappa_B} \Delta H$ integrate $\rightarrow \Delta K = \Delta K(0) + \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B} \Delta H d\tau'$

$K = \int_0^\infty K_\nu d\nu = \int_0^\infty f_\nu J_\nu d\nu = fJ \ , \ H(0) = \int_0^\infty H_\nu(0) d\nu = \int_0^\infty h_\nu J_\nu(0) d\nu = hJ(0)$

$\rightarrow \Delta K = \frac{f(0)\Delta H(0)}{h} + \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B} \Delta H d\tau' = f \Delta J$

$\frac{d\Delta H}{d\tau} = \frac{\kappa_J}{\kappa_B} \Delta J - \Delta B \rightarrow \Delta B = -\frac{d\Delta H}{d\tau} + \frac{\kappa_J}{\kappa_B} \left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f} \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B} \Delta H d\tau' \right)$

$\Delta B = \frac{4\sigma T^3}{\pi} \Delta T = -\frac{dH'}{d\tau} + \frac{dH}{d\tau} + \frac{\kappa_J}{\kappa_B} \left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f} \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B} \Delta H d\tau' \right)$

$$\boxed{\Delta T = \frac{\pi}{4\sigma T^3} \left[\frac{\kappa_J}{\kappa_B} J - B + \frac{\kappa_J}{\kappa_B} \left(\frac{f(0)\Delta H(0)}{fh} + \frac{1}{f} \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B} \Delta H d\tau' \right) \right]}$$

Unsöld-Lucy correction

$$\boxed{\Delta T = \frac{\pi}{4\sigma T^3}\left[\underbrace{\frac{\kappa_J}{\kappa_B}J - B}+\underbrace{\frac{\kappa_J}{\kappa_B}\left(\frac{f(0)\Delta H(0)}{fh}+\frac{1}{f}\int_{\tau'=0}^{\tau}\frac{\kappa_H}{\kappa_B}\Delta H d\tau'\right)}\right]}$$

„Radiative equilibrium" part good at small optical depths but poor at large optical depths $J \to B$

„Flux conservation" part good at large optical depths but poor at small optical depths $\frac{dH}{d\tau} \to 0$

Unsöld-Lucy scheme typically requires damping

Still problems with strong resonance lines, i.e. radiative equilibrium term is dominated by few optically thick frequencies

Stellar Atmospheres: Temperature Correction Schemes

Unsöld-Lucy correction

Generalization for scattering

0-th moment: $\frac{dH_\nu}{dt} = \kappa_\nu\left(J_\nu - S_\nu^{tot}\right) = \kappa_\nu\left(J_\nu - (1-\beta_e)B_\nu - \beta_e J_\nu\right)$

$$\int \cdots d\nu \to \frac{dH}{d\tau} = \frac{\kappa_J}{\kappa_B}J - B$$

$$\kappa_B B = (1-\beta_e)\int_{\nu=0}^{\infty}\kappa_\nu B_\nu d\nu \ , \ \kappa_J J = (1-\beta_e)\int_{\nu=0}^{\infty}\kappa_\nu J_\nu d\nu \ , \ d\tau = \kappa_B dt$$

$\vdots$

All the rest is the same

Difficulties for scattering dominated regions: weak coupling between radiation field and temperature

$$\beta_e \to 1 \Rightarrow \begin{cases} \kappa_B \Delta B \to 0 \\ \kappa_J \Delta J \to 0 \\ \frac{d\Delta H}{d\tau} \to 0 \end{cases}$$

Unsöld-Lucy correction

Generalization to non-LTE (Werner & Dreizler 1998, Dreizler 2003)

0-th moment: $\frac{dH_\nu}{dt} = \kappa_\nu \left(J_\nu - S_\nu \right) = \kappa_\nu J_\nu - \tilde{\kappa}_\nu B_\nu - \gamma_\nu J_\nu$

$$\int \cdots d\nu \rightarrow \frac{dH}{d\tau} = \frac{\kappa_J}{\kappa_B} J - B$$

$$\kappa_B B = \int_{\nu=0}^{\infty} \tilde{\kappa}_\nu B_\nu d\nu \ , \ \kappa_J J = \int_{\nu=0}^{\infty} (\kappa_\nu - \gamma_\nu) J_\nu d\nu \ , \ d\tau = \kappa_B dt$$

$\vdots$

All the rest is the same

$\tilde{\kappa}_\nu$ should contain only terms which couple directly to the temperature, i.e. bf and ff transitions

Depth dependent damping (need to play with parameters **c**):

$$\Delta T = \frac{\pi}{4\sigma T^3} \left[c_1 e^{-\tau_0/\tau} \left(\frac{\kappa_J}{\kappa_B} J - B \right) + \frac{\kappa_J}{\kappa_B} \left(c_2 (1 - e^{-\tau_0/\tau}) \frac{f(0)\Delta H(0)}{fh} + \frac{c_3 (1 - e^{-\tau_0/\tau})}{f} \int_{\tau'=0}^{\tau} \frac{\kappa_H}{\kappa_B} \Delta H d\tau' \right) \right]$$

Stellar Atmospheres

This was the contents of our lecture:

Radiation field
Radiation transfer
Emission and absorption
Radiative equilibrium
Hydrostatic equilibrium
Stellar atmosphere models

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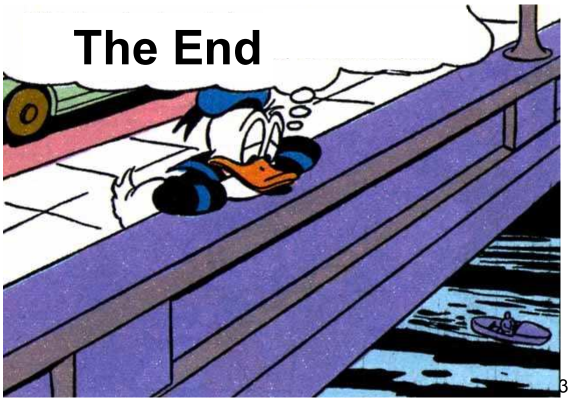

Stellar Atmospheres

This was the contents of our lecture:

Radiation field
Radiation transfer
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Radiative equilibrium
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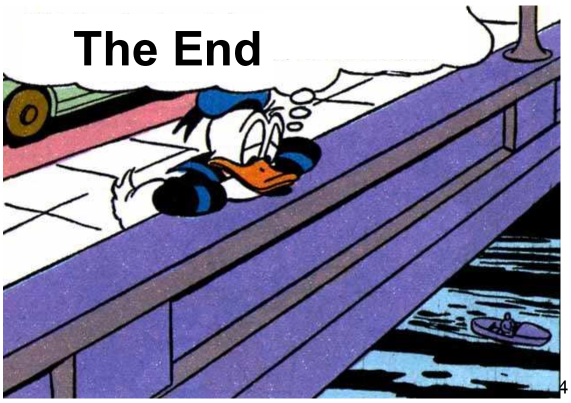
Thank you for listening !



Avrett-Krook method

In case that flux conservation and radiative equilibrium is not fulfilled, Unsöld-Lucy can only change the temperature

Change of other quantities, e.g. opacity, is not accounted for

$\rightarrow$ Avrett & Krook (1963)

strict LTE assumed, generalization straightforward

Current quantities:

$$\mu \frac{dI_\nu^0}{d\tau^0} = \underbrace{\frac{\kappa_\nu^0}{\kappa^0}}_{=\chi_\nu^0} \left(I_\nu^0 - B_\nu^0 (T^0(\tau^0)) \right) \quad \text{with some kind of mean opacity } \kappa^0$$

Does not fulfill flux conservation and radiative equilibrium

New quantities:

$$\mu \frac{dI_\nu}{d\tau} = \underbrace{\frac{\kappa_\nu}{\kappa}}_{=\chi_\nu} \left(I_\nu - B_\nu (T(\tau)) \right) \quad \text{with mean opacity } \kappa$$

Stellar Atmospheres: Temperature Correction Schemes

Avrett-Krook method

Linear Taylor expansion of the new quantities from old ones:

$$\tau = \tau^0 + \tau^1 \rightarrow \frac{d\tau}{d\tau^0} = 1 + \frac{d\tau^1}{d\tau^0} \quad T = T^0 + T^1 \quad I_\nu = I_\nu^0 + I_\nu^1 \quad H = \int (H_\nu^0 + H_\nu^1) d\nu = \sigma / 4\pi T_{\text{eff}}^4$$

$$\chi_\nu = \chi_\nu^0 + \chi_\nu^1 = \chi_\nu^0 + \tau^1 \left. \frac{d\chi_\nu}{d\tau} \right|_0 \quad B_\nu = B_\nu^0 + B_\nu^1 = B_\nu^0 + T^1 \left. \frac{dB_\nu}{dT} \right|_0$$

Radiative transfer equation:

$$\mu \frac{dI_\nu}{d\tau} = \chi_\nu \left(I_\nu - B_\nu (T(\tau)) \right)$$

$$\mu \frac{dI_\nu^0}{d\tau^0} + \mu \frac{dI_\nu^1}{d\tau^0} = \frac{d\tau}{d\tau^0} \left(\chi_\nu^0 + \chi_\nu^1 \right) \left(I_\nu^0 + I_\nu^1 - B_\nu^0 - B_\nu^1 \right)$$

$$\chi_\nu^0 \left(I_\nu^0 - B_\nu^0 \right) + \mu \frac{dI_\nu^1}{d\tau^0} = \left(1 + \frac{d\tau^1}{d\tau^0} \right) \left(\chi_\nu^0 + \chi_\nu^1 \right) \left(I_\nu^0 + I_\nu^1 - B_\nu^0 - B_\nu^1 \right)$$

$$\chi_\nu^0 \left(I_\nu^0 - B_\nu^0 \right) + \mu \frac{dI_\nu^1}{d\tau^0} = \left(\chi_\nu^0 + \chi_\nu^1 + \chi_\nu^0 \frac{d\tau^1}{d\tau^0} \right) \left(I_\nu^0 + I_\nu^1 - B_\nu^0 - B_\nu^1 \right)$$

$$\mu \frac{dI_\nu^1}{d\tau^0} = \chi_\nu^0 \left(I_\nu^1 - B_\nu^1 \right) + \left(\chi_\nu^1 + \chi_\nu^0 \frac{d\tau^1}{d\tau^0} \right) \left(I_\nu^0 - B_\nu^0 \right)$$

Avrett-Krook method

1st moment:

$$\mu \frac{dI_\nu^1}{d\tau^0} = \chi_\nu^0 \left(I_\nu^1 - B_\nu^1\right) + \left(\chi_\nu^1 + \chi_\nu^0 \frac{d\tau^1}{d\tau^0}\right)\left(I_\nu^0 - B_\nu^0\right) \quad \int \cdots \mu \, d\mu$$

$$\rightarrow \frac{dK_\nu^1}{d\tau^0} = \chi_\nu^0 H_\nu^1 + \left(\chi_\nu^1 + \chi_\nu^0 \frac{d\tau^1}{d\tau^0}\right) H_\nu^0 = \chi_\nu^0 H_\nu^1 + \left(\tau^1 \frac{d\chi_\nu}{d\tau^0} + \chi_\nu^0 \frac{d\tau^1}{d\tau^0}\right) H_\nu^0 := 0$$

$$\rightarrow \frac{d\tau^1}{d\tau^0} H_\nu^0 + \tau^1 \frac{d\chi_\nu}{d\tau^0} \frac{H_\nu^0}{\chi_\nu^0} = -H_\nu^1 \quad \int \cdots d\nu$$

$$\rightarrow \frac{d\tau^1}{d\tau^0} H^0 + \tau^1 \int_0^\infty \frac{d\chi_\nu}{d\tau^0} \frac{H_\nu^0}{\chi_\nu^0} d\nu = H^0 - \frac{\sigma}{4\pi} T_{\text{eff}}^4 \quad \text{, linear DEQ of first order}$$

$$\Rightarrow \boxed{\tau^1 = \frac{1}{M(\tau^0)} \int_0^{\tau^0} M(x) \left[1 - \frac{\sigma T_{\text{eff}}^4}{4\pi H^0(x)}\right] dx \quad , \quad M(x) = \exp\left(\int_0^x dy \int_0^\infty d\nu \frac{1}{\chi_\nu^0} \frac{d\chi_\nu}{d\tau^0} \frac{H_\nu^0}{H^0}\right)}$$

Avrett-Krook method

Outer boundary:

$$H_\nu^1(0) = h_\nu J_\nu^1(0) \quad \int \cdots d\nu$$

$$H^1(0) = \frac{\sigma}{4\pi} T_{\text{eff}}^4 - H^0(0) = \left(\frac{\sigma T_{\text{eff}}^4}{4\pi H^0(0)} - 1\right) H^0(0) = \left(\frac{\sigma T_{\text{eff}}^4}{4\pi H^0(0)} - 1\right) \int_0^\infty H_\nu^0(0) d\nu$$

$$\rightarrow \int_0^\infty \left(\frac{\sigma T_{\text{eff}}^4}{4\pi H^0(0)} - 1\right) H_\nu^0(0) d\nu = \int_0^\infty h_\nu J_\nu^1(0) d\nu$$

$$\rightarrow J_\nu^1(0) = \left(\frac{\sigma T_{\text{eff}}^4}{4\pi H^0(0)} - 1\right) \frac{H_\nu^0(0)}{h_\nu}$$

$$\frac{dK_\nu^1}{d\tau^0} := 0 \Rightarrow f_\nu J_\nu^1(\tau^0) = \text{const} = f_\nu J_\nu^1(0) = \left(\frac{\sigma T_{\text{eff}}^4}{4\pi H^0(0)} - 1\right) \frac{H_\nu^0(0)}{h_\nu}$$

$$\Rightarrow \boxed{J_\nu^1(\tau^0) = \left(\frac{\sigma T_{\text{eff}}^4}{4\pi H^0(0)} - 1\right) \frac{H_\nu^0(0)}{f_\nu h_\nu}}$$

Avrett-Krook method

0-th moment:

$$\mu \frac{dI_\nu^1}{d\tau^0} = \chi_\nu^0\left(I_\nu^1 - B_\nu^1\right) + \left(\chi_\nu^1 + \chi_\nu^0 \frac{d\tau^1}{d\tau^0}\right)\left(I_\nu^0 - B_\nu^0\right) \quad \int \cdots d\mu$$

$$\rightarrow \frac{dH_\nu^1}{d\tau^0} = \chi_\nu^0\left(J_\nu^1 - B_\nu^1\right) + \left(\chi_\nu^1 + \chi_\nu^0 \frac{d\tau^1}{d\tau^0}\right)\left(J_\nu^0 - B_\nu^0\right)$$

$$\rightarrow \frac{dH_\nu^1}{d\tau^0} = \chi_\nu^0\left(J_\nu^1 - T^1 \frac{dB_\nu}{dT^0}\right) + \left(\tau^1 \frac{d\chi_\nu}{d\tau^0} + \chi_\nu^0 \frac{d\tau^1}{d\tau^0}\right)\left(J_\nu^0 - B_\nu^0\right) \quad \int \cdots d\nu$$

$$\rightarrow \frac{dH^1}{d\tau^0} = -\frac{dH^0}{d\tau^0} = \int_0^\infty \chi_\nu^0 J_\nu^1 d\nu - T^1 \int_0^\infty \chi_\nu^0 \frac{dB_\nu}{dT^0} d\nu + \tau^1 \int_0^\infty \frac{d\chi_\nu}{d\tau^0}\left(J_\nu^0 - B_\nu^0\right) d\nu + \frac{d\tau^1}{d\tau^0}\frac{dH^0}{d\tau^0}$$

$$\Rightarrow \boxed{T^1 = 1 \Big/ \int_0^\infty \chi_\nu^0 \frac{dB_\nu}{dT^0} d\nu \left[\tau^1 \int_0^\infty \frac{d\chi_\nu}{d\tau^0}\left(J_\nu^0 - B_\nu^0\right) d\nu + \left(1 + \frac{d\tau^1}{d\tau^0}\right)\frac{dH^0}{d\tau^0} + \left(\frac{\sigma T_{\mathrm{eff}}^4}{4\pi H^0(0)} - 1\right)\int_0^\infty \chi_\nu^0 \frac{H_\nu^0(0)}{f_\nu h_\nu} d\nu\right]}$$

Radiative equilibrium and Complete Linearization (LTE)

Simultaneous solution of RT and RE radiation transfer:

$$J_\nu(\nu,\tau) - \frac{d^2 J_\nu(\nu,\tau)}{d\tau(\nu)^2} - B_\nu(\nu, T(\tau)) = 0$$

$$\rightarrow f_{ik}(\vec{J}_k, \vec{T}) = 0 \quad \vec{J}_k = \left(J_{1,k}, \cdots, J_{i,k}, \cdots, J_{ND,k}\right) \quad \vec{T} = (T_1, \cdots, T_i, \cdots, T_{ND})$$

$$\vec{J}_k^0, \vec{T}^0 \quad f_{ik}(\vec{J}_k^0, \vec{T}^0) \neq 0 \Rightarrow \text{correction } \delta\vec{J}_k, \delta\vec{T} \rightarrow f_{ik}(\vec{J}_k^0 + \delta\vec{J}_k, \vec{T}^0 + \delta\vec{T}) = 0$$

$$\text{Taylor expansion: } f_{ik}(\vec{J}_k^0 + \delta\vec{J}_k, T_i^0 + \delta T)$$

$$= 0 = f_{ik}(\vec{J}_k^0, \vec{T}^0) + \frac{\partial f_{ik}}{\partial J_{i-1k}}\delta J_{i-1k} + \frac{\partial f_{ik}}{\partial J_{ik}}\delta J_{ik} + \frac{\partial f_{ik}}{\partial J_{i+1k}}\delta J_{i+1k} + \frac{\partial f_{ik}}{\partial T_i}\delta T_i$$

$$\rightarrow T_k \delta\vec{J}_k + U_k \delta\vec{T} = \vec{K}_k$$

T_k : tri-diagonal with usual $-A_{ik}, B_{ik}, -C_{ik}$

$$U_k: \text{ diagonal } \left(U_k\right)_{ii} = -\frac{\partial B_\nu(\nu_k, T_i)}{\partial T_i}$$

$$\left(\vec{K}_k\right)_i = -f_{ik}(\vec{J}_k^0, \vec{T}^0)$$

Radiative equilibrium and Complete Linearization (LTE)

Simultaneous solution of RT and RE radiative equilibrium:

$$\int_{\nu=0}^{\infty} \kappa(\nu,\tau_i)\left(J_\nu(\nu,\tau_i) - B_\nu(\nu,T_i)\right) d\nu = 0$$

$$\to f_{i,NF+1}\left(J_{i,1},\cdots,J_{i,k},\cdots,J_{i,NF},T_i\right) = \sum_{k=1}^{NF} w_k \left(J_{ik} - B_\nu(\nu_k,T_i)\right) = 0$$

Taylor expansion: $f_{i,NF+1}\left(\vec{J}_k^0 + \delta\vec{J}_k, T_i^0 + \delta T\right)_{k=1,NF}$

$$= 0 = f_{i,NF+1}(\vec{J}_k^0, \vec{T}^0) + \sum_{k=1}^{NF} \frac{\partial f_{i,NF+1}}{\partial J_{ik}} \delta J_{ik} + \frac{\partial f_{i,NF+1}}{\partial T_i} \delta T_i \to \sum_{\mathrm{k}=1}^{\mathrm{NF}} W_k \delta\vec{J}_k + D\delta\vec{T} = \vec{L}$$

W_k : diagonal $\left(W_k\right)_{ii} = w_k$

D: diagonal $\left(D\right)_{ii} = -\sum_{k'=1}^{NF} w_{k'} \dfrac{\partial B_\nu(\nu_{k'},T_i)}{\partial T_i}$

$$\left(\vec{L}\right)_i = -\sum_{k=1}^{NF} w_k \left(J_{ik}^0 - B_\nu(\nu_k,T_i^0)\right)$$

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Radiative equilibrium and Complete Linearization (LTE)

Together: Rybicki scheme:

$$\left(\begin{array}{ccccc|c} T_1 & & & & & U_1 \\ & \ddots & & 0 & & \vdots \\ & & T_k & & & U_k \\ & 0 & & \ddots & & \vdots \\ & & & & T_{NF} & U_{NF} \\ \hline W_1 & & W_k & & W_{NF} & D \end{array}\right) \left[\begin{array}{c} \delta\vec{J}_1 \\ \vdots \\ \delta\vec{J}_k \\ \vdots \\ \delta\vec{J}_{NF} \\ \hline \delta\vec{T} \end{array}\right] = \left[\begin{array}{c} \vec{K}_1 \\ \vdots \\ \vec{K}_k \\ \vdots \\ \vec{K}_{NF} \\ \hline \vec{L} \end{array}\right]$$

RE takes the part of the scattering integral

Instead of $\vec{J}$ solve for temperature corrections

Non-linear $\to$ iteration

During the iteration: new opacities, Eddington factors

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Radiative equilibrium and Complete Linearization (NLTE)

Direct generalization at least problematic due to weak coupling of NLTE source function to the temperature

Take into account the change of the population numbers

$\rightarrow$ Add RE to the Complete Linearization scheme (Auer & Mihalas 1969)

Radiative equilibrium in NLTE:

$$\int_{\nu=0}^{\infty} \kappa(\nu,\tau_i)\left(J_\nu(\nu,\tau_i)-S_\nu(\nu,\tau_i)\right)d\nu=0$$

$$\rightarrow f_{i,NF+NL+1}\left(J_{i,1},\cdots,J_{i,NF},n_{i,1},\cdots,n_{i,NL},T_i\right)=\sum_{k=1}^{NF} w_k\left(\kappa(\nu_k,\tau_i)J_{ik}-\eta_\nu(\nu_k,\tau_i)\right)=0$$

Linearization: $f_{i,NF+NL+1}(\vec{\psi}^0+\delta\vec{\psi})=$

$$w_k\left(J_{ik}\frac{\partial\kappa_k}{\partial n_{il}}-\frac{\partial\eta_k}{\partial n_{il}}\right)$$

$$0=f_{i,NF+NL+1}(\vec{\psi}^0)+\sum_{k=1}^{NF}\frac{\partial f_{i,NF+NL+1}}{\partial J_{ik}}\delta J_{ik}+\sum_{l=1}^{NL}\frac{\partial f_{i,NF+NL+1}}{\partial n_{il}}\delta n_{il}+\frac{\partial f_{i,NF+NL+1}}{\partial T_i}\delta T_i$$

Radiative equilibrium and Complete Linearization (NLTE)

Together with RT and SE:

$$f_{i,\alpha}(\psi^0+\delta\psi)=\qquad \alpha=1\cdots NF+NL$$

$$0=f_{i,\alpha}(\psi^0)+\sum_{i=1}^{ND}\left\{\sum_{k=1}^{NF}\frac{\partial f_{i,\alpha}}{\partial J_{i,k}}\delta J_{i,k}+\sum_{l=1}^{NL}\frac{\partial f_{i,\alpha}}{\partial n_{i,l}}\delta n_{i,l}+\frac{\partial f_{i,\alpha}}{\partial T_i}\delta T_i\right\}$$

$$-A_i\delta\vec{\psi}_{i-1}+B_i\delta\vec{\psi}_i-C_i\delta\vec{\psi}_{i+1}=\vec{L}_i$$

$$-\begin{pmatrix}\ddots & 0 & & \\ A_{i,k} & & 0 & \\ 0 & \ddots & & \\ & & & \\ 0 & & 0 & \\ & & & \\ 0 & & 0 & \end{pmatrix}\begin{bmatrix}\delta\vec{J}_{i-1}\\ \\ \\ \delta\vec{n}_{i-1}\\ \\ \delta T_{i-1}\end{bmatrix}+\begin{pmatrix}\ddots & 0 & & & \\ & B_{i,k} & & & \\ 0 & & \ddots & & \\ & & & & \\ & & & & \end{pmatrix}\begin{bmatrix}\delta\vec{J}_i\\ \\ \\ \delta\vec{n}_i\\ \\ \delta T_i\end{bmatrix}-\begin{pmatrix}\ddots & 0 & & \\ & C_{i,k} & & 0\\ 0 & & \ddots & \\ & & & \\ & 0 & & 0\\ & & & \\ & 0 & & 0\end{pmatrix}\begin{bmatrix}\delta\vec{J}_{i+1}\\ \\ \\ \delta\vec{n}_{i+1}\\ \\ \delta T_i\end{bmatrix}=-\begin{bmatrix}\\ \\ \\ f_{i,\alpha}(\psi^0)\\ \\ \\ \end{bmatrix}$$

Radiative equilibrium and Complete Linearization (NLTE)

Matrix B_i:

$$
B_i = \begin{array}{cc}
\begin{array}{ccccccc} 1 & \dots & NF & 1 & \dots & NL & T \end{array} & \\
\left(\begin{array}{cccc|cc|c}
\ddots & & 0 & & \vdots & & \\
 & B_{i,k} & & \cdots & -\frac{\partial S_{i,k}}{\partial n_{i,l'}} & \cdots & -\frac{\partial S_{i,k}}{\partial T_i} \\
0 & & \ddots & & \vdots & & \\
 & \vdots & & & \vdots & & \\
\hline
\cdots & \sum_{m=1}^{NL} \frac{\partial (P_i)_{l,m}}{\partial J_{i,k'}} n_{i,m} & \cdots & \cdots & (P_i)_{l,l'} & \cdots & \frac{\partial (P_i)_{l,m}}{\partial T_i} T_i \\
 & \vdots & & & \vdots & & \\
\hline
 & \frac{\partial f_{iNF+NL+1}}{\partial J_{i,k'}} & & & \frac{\partial f_{iNF+NL+1}}{\partial n_{i,l'}} & & \frac{\partial f_{iNF+NL+1}}{\partial T_i}
\end{array}\right) &
\begin{array}{c} 1 \\ \dots NF \\ 1 \\ \dots NL \\ T \end{array}
\end{array}
$$

Accelerated Lambda Iteration

Motivation

Complete Linearization provides a solution scheme, solving the radiation transfer, the statistical equilibrium and the radiative equilibrium simultaneously.

But, the system is coupled over all depths (via RT) and all frequencies (via SE, RE) → HUGE!

Abbreviations used in this chapter:

RT = Radiation Transfer equations

SE = Statistical Equilibrium equations

RE = Radiative Equilibrium equation

Multi-frequency / multi-gray

Ways around:

Multi-frequency / multi-gray method by Anderson (1985,1989)

- Group all frequency points according to their opacity into bins (typically 5) and solve the RT with mean opacities of these bins. → Only 5 RT equations instead of thousands
- Use a Complete Linearization with the reduced set of equations
- Solve RT alone in between to get all intensities, Eddington-factors, etc.
- Main disadvantage: in principle depth dependent grouping

Lambda Iteration

Split RT and SE+RE:

$$J^{new} = \Lambda S^{old}(n,T)$$ RT formal solution

$$\underline{\underline{A(J,T)}}\,\underline{n}^{new} = \underline{b}$$ SE

$$\int_0^\infty \kappa(\nu,n,T)\left(J_\nu - S_\nu(\nu,n,T)\right) d\nu = 0$$ RE

- Good: SE is linear (if a separate T-correction scheme is used)
- Bad: SE contain old values of n,T (in rate matrix A)

Disadvantage: not converging, this is a Lambda iteration!

Accelerated Lambda Iteration (ALI)

Again: split RT and SE+RE but now use ALI

$$J^{new} = \Lambda S^{old}(n^{old}, T^{old}) + \Lambda^* S^{new}(n^{new}, T^{new}) - \Lambda^* S^{old}(n^{old}, T^{old}) \quad \text{RT}$$

$$\underline{\underline{A}}(J^{new}, T^{new})\underline{n}^{new} = \underline{b} \quad \text{SE}$$

$$\int_0^\infty \kappa(\nu, n^{new}, T^{new})\left(J_\nu^{new} - S_\nu(\nu, n^{new}, T^{new})\right) d\nu = 0 \quad \text{RE}$$

- Good: SE contains new quantities n, T
- Bad: Non-Linear equations → linearization (but without RT)

Basic advantage over Lambda Iteration: ALI converges!

Example: ALI working on Thomson scattering problem

$S = (1-\beta_e)B + \beta_e J$ source function with scattering, problem: J unknown→iterate

$$\Rightarrow J^{new} = (\Lambda - \Lambda^*) S^{old} + \Lambda^* S^{new}$$

$$= (\Lambda - \Lambda^*) S^{old} + \Lambda^*\left((1-\beta_e)B^{new} + \beta_e J^{new}\right) \qquad J^{FS} := \text{formal solution on } S^{old}$$

$$= J^{FS} - \Lambda^*\left((1-\beta_e)B^{old} + \beta_e J^{old} - (1-\beta_e)B^{new} - \beta_e J^{new}\right) \qquad B^{old} = B^{new}$$

$$= J^{FS} - \Lambda^*\left(\beta_e J^{old} - \beta_e J^{new}\right) \qquad \text{solve for } J^{new}$$

$$\Rightarrow J^{new} = \left[1 - \Lambda^* \beta_e\right]^{-1}\left(J^{FS} - \Lambda^* \beta_e J^{old}\right) \qquad \text{subtract } J^{old} \text{ on both sides}$$

$$\Rightarrow J^{new} - J^{old} = \left[1 - \Lambda^* \beta_e\right]^{-1}\left(J^{FS} - J^{old}\right)$$

amplification factor

Interpretation: iteration is driven by difference (J^{FS}-J^{old}) but: this difference is amplified, hence, iteration is accelerated.
Example: β_e=0.99; at large optical depth Λ^* almost 1 → strong amplifaction

What is a good Λ*?

The choice of Λ* is in principle irrelevant but in practice it decides about the success/failure of the iteration scheme.

First (useful) Λ* (Werner & Husfeld 1985):

$$\Lambda^*_\nu(\tau,\tau')\,S_\nu(\tau') = \begin{cases} S_\nu(\tau) & \tau > \gamma \\ 0 & \tau \le \gamma \end{cases}$$

A few other, more elaborate suggestions until Olson & Kunasz (1987): Best Λ* is the diagonal of the Λ-matrix

(Λ-matrix is the numerical representation of the integral operator Λ)

We therefore need an efficient method to calculate the elements of the Λ-matrix (are essentially functions of τ_ν).

Could compute directly elements representing the Λ-integral operator, but too expensive (E_1 functions). Instead: use solution method for transfer equation in differential (not integral) form: short characteristics method

In the final lecture tomorrow, we will learn two important methods to obtain numerically the formal solution of the radiation transfer equation.

1. Solution of the differential equation as a boundary-value problem (Feautrier method). [can include scattering]

2. Solution employing Schwarzschild equation on local scale (short characteristics method). [cannot include scattering, must ALI iterate]

The direct numerical evaluation of Schwarzschild equation is much too cpu-time consuming, but in principle possible.

Olson-Kunasz Λ*

Short characteristics with linear approximation of source function

$$I^+(\tau,\mu,\nu) = I^+(\tau_{\max},\mu,\nu)\exp\left(-\frac{\tau_{\max}-\tau}{\mu}\right) + \int_{\tau}^{\tau_{\max}} S(\tau')\exp\left(-\frac{\tau'-\tau}{\mu}\right)\frac{d\tau'}{\mu}$$

$$I^-(\tau,\mu,\nu) = I^-(0,\mu,\nu) \quad \exp\left(-\frac{\tau}{|\mu|}\right) \qquad + \int_{0}^{\tau} S(\tau')\exp\left(-\frac{\tau-\tau'}{|\mu|}\right)\frac{d\tau'}{|\mu|}$$

$$I^+(\tau_i,\mu,\nu) = I^+(\tau_{i+1},\mu,\nu)\exp(-\Delta\tau_i) + \Delta I_i^+(S,\mu,\nu)$$

$$I^-(\tau_i,\mu,\nu) = I^-(\tau_{i-1},\mu,\nu)\exp(-\Delta\tau_{i-1}) + \Delta I_i^-(S,\mu,\nu)$$

with $\Delta\tau_{i-1} = \frac{(\tau_i - \tau_{i-1})}{|\mu|}$

using a linear interpolation for the spatial variation of S

the intergrals $\Delta I_i^\pm$ can be evaluated as

$$\Delta I_i^\pm = \alpha_i^\pm S_{i-1} + \beta_i^\pm S_i + \gamma_i^\pm S_{i+1}$$

Olson-Kunasz Λ*

Short characteristics with linear approximation of source function

$$\alpha_i^+ = 0 \qquad \alpha_i^- = -e^{-\Delta} + \frac{1-e^{-\Delta}}{\Delta}$$

$$\beta_i^+ = 1 + \frac{e^{-\Delta}-1}{\Delta} \qquad \beta_i^- = 1 - \frac{1-e^{-\Delta}}{\Delta}$$

$$\gamma_i^+ = -e^{-\Delta} - \frac{e^{-\Delta}-1}{\Delta} \qquad \gamma_i^- = 0$$

$$J = \frac{1}{2}\int_0^1 \left(I^+ + I^-\right) d\mu = \frac{1}{2}\int_0^1 \left(\Lambda_\mu^+ S + \Lambda_\mu^- S\right) d\mu$$

use $S = (0,\cdots,1,\cdots,0)^T$ for $(0,\cdots,i,\cdots,0)$ to project colums of Λ

Inward

$$\begin{array}{c} i \\ \begin{pmatrix} & \vdots & \\ & 0 & \\ & \vdots & \\ & \Delta\hat{I}^-_{i-1} & \\ 0 & \hat{I}^-_{i-1}\exp(-\Delta\tau_{i-1})+\Delta\hat{I}^-_i & 0 \\ & \hat{I}^-_i\exp(-\Delta\tau_i)+\Delta\hat{I}^-_{i+1} & \\ & \vdots & \\ & \hat{I}^-_{k=i+2\cdots ND} & \\ & \vdots & \end{pmatrix} \end{array} =$$

$$\begin{array}{c} i \\ \begin{pmatrix} & \vdots & \\ & 0 & \\ & \vdots & \\ & \gamma^-_{i-1} & \\ 0 & \gamma^-_{i-1}\exp(-\Delta\tau_{i-1})+\beta^-_i & 0 \\ & \left(\gamma^-_{i-1}\exp(-\Delta\tau_{i-1})+\beta^-_i\right)\exp(-\Delta\tau_i)+\alpha^-_{i+1} & \\ & \vdots & \\ & \hat{I}^-_{k-1}\exp(-\Delta\tau_{k-1}) & \\ & \vdots & \end{pmatrix} \end{array} = \begin{array}{c} i \\ \begin{pmatrix} & \vdots & \\ & 0 & \\ & \vdots & \\ & 0 & \\ 0 & \beta^-_i & 0 \\ & \beta^-_i\exp(-\Delta\tau_i)+\alpha^-_{i+1} & \\ & \vdots & \\ & \hat{I}^-_{k-1}\exp(-\Delta\tau_{k-1}) & \\ & \vdots & \end{pmatrix} \end{array}$$

Outward

$$\begin{array}{c} i \\ \begin{pmatrix} & \vdots & \\ & \hat{I}^+_{k=0\cdots i-2} & \\ & \vdots & \\ & \hat{I}^+_i\exp(-\Delta\tau_{i-1})+\Delta\hat{I}^+_{i-1} & \\ 0 & \hat{I}^+_{i+1}\exp(-\Delta\tau_i)+\Delta\hat{I}^+_i & 0 \\ & \Delta\hat{I}^+_{i+1} & \\ & \vdots & \\ & 0 & \\ & \vdots & \end{pmatrix} \end{array} =$$

$$\begin{array}{c} i \\ \begin{pmatrix} & \vdots & \\ & \hat{I}^+_{k+1}\exp(-\Delta\tau_{k-1}) & \\ & \vdots & \\ & \left(\alpha^+_{i+1}\exp(-\Delta\tau_i)+\beta^+_i\right)\exp(-\Delta\tau_{i-1})+\gamma^+_{i-1} & \\ 0 & \alpha^+_{i+1}\exp(-\Delta\tau_i)+\beta^+_i & 0 \\ & \alpha^+_{i+1} & \\ & \vdots & \\ & 0 & \\ & \vdots & \end{pmatrix} \end{array} = \begin{array}{c} i \\ \begin{pmatrix} & \vdots & \\ & \hat{I}^+_{k+1}\exp(-\Delta\tau_{k-1}) & \\ & \vdots & \\ & \beta^+_i\exp(-\Delta\tau_{i-1})+\gamma^+_{i-1} & \\ 0 & \beta^+_i & 0 \\ & 0 & \\ & \vdots & \\ & 0 & \\ & \vdots & \end{pmatrix} \end{array}$$

Λ-Matrix

$$\Lambda_{*,i} = \begin{pmatrix} & \vdots & \\ & \frac{1}{2}\int_0^1 d\mu \hat{I}^{+}_{k+1} \exp(-\Delta\tau_{k-1}) & \\ & \vdots & \\ & \frac{1}{2}\int_0^1 d\mu \left(\beta_i^{+} \exp(-\Delta\tau_{i-1}) + \gamma_{i-1}^{+}\right) & \\ 0 & \frac{1}{2}\int_0^1 d\mu \left(\beta_i^{+} + \beta_i^{-}\right) & 0 \\ & \frac{1}{2}\int_0^1 d\mu \left(\beta_i^{-} \exp(-\Delta\tau_{i}) + \alpha_{i+1}^{-}\right) & \\ & \vdots & \\ & \frac{1}{2}\int_0^1 d\mu \left(\hat{I}^{-}_{k-1} \exp(-\Delta\tau_{k-1})\right) & \\ & \vdots & \end{pmatrix}$$

Towards a linear scheme

Λ* acts on S, which makes the equations non-linear in the occupation numbers

- Idea of Rybicki & Hummer (1992): use $J=\Delta J+\Psi^{*}\eta^{new}$ instead
- Modify the rate equations slightly:

$$R_{ij} n_i = 4\pi \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_i J_\nu d\nu = 4\pi \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_i \left(\Psi^{*}\eta(n) + \Delta J\right) d\nu$$

$$R_{ji} n_j = 4\pi \left(\frac{n_i}{n_j}\right)^{*} \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_j \left(J_\nu + \frac{2h\nu^3}{c^2}\right) d\nu$$

$$= 4\pi \left(\frac{n_i}{n_j}\right)^{*} \int_0^\infty \frac{\sigma_{ij}}{h\nu} n_j \left(\Psi^{*}\eta(n) + \Delta J + \frac{2h\nu^3}{c^2}\right) d\nu$$