Literature

- Bowers, R., Deeming, T.
  Astrophysics I - Stars
  Jones & Bartlett, Boston 1984

- Kippenhahn, R., Weigert, A.
  Stellar Structure and Evolution
  Springer, Berlin 1994

- Chandrasekhar, S.
  An Introduction to the Study of
  Stellar Structure
  Dover, New York 1939/1967

- Clayton, D.D.
  Principles of Stellar Evolution and
  Nucleosynthesis

- Cox, J.P., Giuli, R.T.
  Principles of Stellar Structure, vol. 1 and 2
  Gordon & Breach, New York 1968

- Shapiro, S.L., Teukolsky, S.A.
  Black Holes, White Dwarf, and Neutron stars
  Wiley, New York 1983
Information on astronomical objects

- Electromagnetic radiation

- Cosmic radiation (O, Supernovae)
  
  90% protons
  10% $\alpha$-particles
  
  Heavy nuclei
  
  energies: $10^8$ .... $10^{20}$ eV
  
  (laboratory: $10^{12}$ eV)

  diffuse radiation

- Neutrinos (O, Supernovae)

  Detectors: Chlorine
                  Gallium
                  Water

- Gravitational radiation

  indirect: binary pulsar
  
  direct: ?

  (Weber 1960 ?)
Description of radiation field

Intensity \( I_r \)

\[
I_r = \frac{dE_r}{dt \, dv \, dw \, (\hat{n} \cdot df)}
\]

Energy / time / frequency / solid angle / area \( \parallel \hat{n} \)

\[
\hat{n} \cdot df = \cos \Theta \, |df| \\
\, dw = \sin \Theta \, d\Theta \, d\phi \\
\Theta \in (0, \pi) \\
\phi \in (0, 2\pi)
\]

Angular dependence replaced by description by moments:

\[
\int I_r \cos^m \Theta \, dw
\]
0th order moment

\[ M_0 = \frac{1}{4\pi} \int I_v \, dv \omega \]  

mean intensity

physical meaning: consider energy density

\[ U_v = \frac{dE_v}{dv \omega \, d\Omega} = \frac{dE_v}{dv \, d\Omega \, d\Phi} \]

\[ = \frac{1}{c} \int I_v \, dv \omega = \frac{c}{e} \, \gamma_v \]

\[ U_v = \frac{c}{e} \, \gamma_v \]

1st order moment

\[ F_v = \left[ \frac{4}{15} \right] \int I_v \cos \Theta \, d\omega \]

radiation flux

splitting into two parts:

\[ F_v = \int_{\Omega} d\Phi \int_0^{\pi/2} \sin \Theta \, \cos \Theta \, I_v(\Theta, \Phi) \, d\Theta \]

\[ = \int_0^{\pi/2} d\Theta \, F_v^+ + \int_{\pi/2}^{\pi} d\Theta \, F_v^- \]
2nd order moment

\[ K_\nu = \frac{\lambda}{4\pi} \int I_\nu \cos^2 \Theta \, d\omega \]

physical meaning: consider pressure

\[ \sim \text{photon momentum} \frac{d\phi}{\text{time/area}} \]

\[ \sim \frac{\lambda}{c} \text{photon energy} \frac{dE_\nu}{dt \, d\nu \, d\Omega} \]

\[ \sim \frac{\lambda}{c} \cos \Theta \frac{dE_\nu}{dt \, d\nu} \frac{d\Omega}{d\Omega} \]

\[ \sim \frac{\lambda}{c} \cos \Theta \, I_\nu \, \cos \Theta \, d\omega \]

\[ P_\nu = \frac{\lambda}{c} \int I_\nu \cos^2 \Theta \, d\omega \]

\[ P_\nu = \frac{\lambda I_\nu}{c} \]

Isotropic radiation field:

\[ \gamma_\nu = I_\nu \quad F_\nu = 0 \quad K_\nu = \frac{\lambda}{3} I_\nu \quad P_\nu = \frac{\lambda}{3} I_\nu \]

\[ \nu \text{-dependence not relevant: } \int \frac{d\Omega}{d\Omega} \]

\[ I = \int I_\nu \, d\nu \]

For \( \gamma \)-dependence, common definition:

\[ \mu = \cos \Theta \]

\[ e.g.: \quad K_\nu = \frac{\lambda}{4\pi} \int I_\nu \cos^2 \Theta \sin \Theta \, d\omega \, d\phi \]

\[ = \frac{\lambda}{c} \int I_\nu (\mu) \mu^2 \, d\mu \]
Intra-grain scattering, radial field, radiative field

\[ \text{change } dI_r \text{ along } ds : \]
\[ dI_r = \frac{dI_r}{ds} \bigg|_{\text{absorption}} ds + \frac{dI_r}{ds} \bigg|_{\text{scattering}} ds + \frac{dI_r}{ds} \bigg|_{\text{emission}} ds \]

\[ \frac{dI_r}{ds} \bigg|_{\text{absorption}} = -I_r \kappa_r^a \]
\[ \kappa_r^a \text{: volume opacity (cross section/volume)} \]

\[ \frac{dI_r}{ds} \bigg|_{\text{emission}} = \dot{I}_r \text{ emissivity} \]

scattering: contributes both to opacity and emissivity formally

Including scattering in \( \kappa_r^a \) and \( \dot{I}_r \):

\[ \frac{dI_r}{ds} = -I_r \kappa_r^a + \dot{I}_r \]

Specific opacity \( \kappa_r \):

\[ \kappa_r = \kappa_r^a / \rho \text{ (cross section/mass)} \]
Definition: optical depth $\tau_v$

$$d\tau_v = k_v ds$$

$\tau_v$ = photon geometrical distance to mean free path

So far, $ds \parallel d\tau_v$ assumed implicitly

Consider now

$$-dx = \cos \theta \, ds$$

$$-\cos \theta \frac{dI_v}{dx} = -k_v I_v + j_v$$

Equations for radiative transport:

$$\cos \theta \frac{dI_v}{k_v dx} = I_v - \frac{j_v}{k_v}$$

$$\mu \frac{dI_v}{d\tau_v} = I_v - S_v$$

with

$$\mu = \cos \theta \quad ; \quad S_v = \frac{j_v}{k_v} \quad \text{source function}$$

$$d\tau_v = k_v dx \quad \text{optical depth}$$
Radiative transport using moments of the radiative field

\[ \cos \theta \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \]

0th order moment:
\[ \frac{\lambda}{\epsilon_\nu} \int d\omega \]

\[ \frac{d}{d\tau_\nu} \frac{\lambda}{\epsilon_\nu} F_\nu = \gamma_\nu - \frac{\lambda}{\epsilon_\nu} \int S_\nu d\omega \]

\( S_\nu \) isotropic:
\[ \frac{\lambda}{\epsilon_\nu} \frac{dF_\nu}{d\tau_\nu} = \gamma_\nu - S_\nu \]

1st order moment:
\[ \frac{\lambda}{\epsilon_\nu} \int \cos \theta d\omega \]

\[ \frac{dK_\nu}{d\tau_\nu} = \frac{\lambda}{\epsilon_\nu} F_\nu - \frac{\lambda}{\epsilon_\nu} \int \cos \theta S_\nu d\omega \]

\( S_\nu \) isotropic:
\[ \frac{dK_\nu}{d\tau_\nu} = \frac{\lambda}{\epsilon_\nu} F_\nu \]

\[ \frac{dP_\nu}{d\tau_\nu} = \frac{1}{c} F_\nu \]

Stellar criterion: Frequency dependence ignored, \( \nu \)-integration
\[ F = \int F_v \, dv \quad \text{Prad} = \int P_v \, dv \]

\[ F_v = -\frac{c}{k_v} \frac{dp_r}{dr} \quad ; \quad F = -\frac{c}{k_e} \frac{d\text{Prad}}{dr} \]

Relation \( k_v \leftrightarrow k_e \) ?

Definition of the mean \( \bar{k} \) ?
Local thermodynamic equilibrium

**Assumptions:**

1) Local isotropy of radiation field
2) Source function $S_\nu$ corresponds to that of black body $B_\nu(T)$

System big enough for thermodynamic equilibrium and definition of temperature small enough for isotropy to be reasonable approximation in certain respects

**Globally:** No thermodynamic equilibrium temperature $T$; flux does not vanish

**Consequences:**

$$S_\nu = B_\nu(T) = \frac{2\hbar \nu^3}{c^2} \frac{\lambda}{\exp(h\nu/kT) - 1}$$

$$B(T) = \int B_\nu d\nu = \frac{\sigma}{\epsilon} T^4 \quad ; \quad \sigma = \frac{2\pi^2}{15}$$

$S_\nu$ isotropic and $\frac{dF_\nu}{dC_\nu} = 0 \quad [F_\nu \neq 0 \text{ globally}]$

$$\chi_\nu = S_\nu \quad ; \quad \chi_\nu = I_\nu \quad ; \quad \frac{dK_\nu}{dC_\nu} = \frac{\lambda}{4\pi} F_\nu$$

$$\frac{dF_\nu}{dC_\nu} = \frac{\Delta}{c} F_\nu$$
\[ I_\nu = S_\nu = B_\nu (T) \]

\[ p_\nu = \frac{4\pi}{c} k_\nu = \frac{4\pi}{c} \frac{A}{3} I_\nu = \frac{4\pi}{3c} B_\nu \]

\[ u_\nu = 3 p_\nu = \frac{4\pi}{c} I_\nu \]

\[ P_{\text{rad}} = \frac{a}{3} T^4 \quad \text{and} \quad u_{\text{rad}} = a T^4 \]

\textbf{Opacity mean}

\[ F = \int E_\nu d\nu = - \int d\nu \frac{c}{k_\nu} \frac{dP_\nu}{dr} \]

\[ = - \int d\nu \frac{c}{k_\nu} \frac{4\pi}{3c} \frac{dB_\nu}{dT} \frac{dT}{dr} \]

\[ \frac{dP_{\text{rad}}}{dr} = \int d\nu \frac{dP_\nu}{dr} = \int d\nu \frac{4\pi}{3c} \frac{dB_\nu}{dT} \frac{dT}{dr} \]

\[ F = - \frac{c}{k_\nu} \frac{dP_{\text{rad}}}{dr} \]

\[ \overline{\nu} = \frac{\int_0^\infty d\nu \frac{1}{k_\nu} \frac{dB_\nu}{dT} \frac{dT}{dr}}{\int_0^\infty d\nu \frac{dB_\nu}{dT}} \]

\[ \overline{\kappa} : \text{Rosseland mean of opacity} \]

\[ F = - \frac{4ac}{3} T^3 \frac{\lambda}{k_\nu} \frac{dT}{dr} \]

\textbf{Diffusion equation for radiative transport}

"Diffusion approximation" [stellar interiors]
Definition: Luminosity $L$

\[ L(r) = 4\pi r^2 F \]

\[ L(r) = -\frac{16\pi abc}{3h} r^2 T^3 \frac{dT}{dr} \]

Alternative forms:

\[ F = -\frac{1}{3} \frac{c}{h} \frac{du_{\text{rad}}}{dr} \]

\[ -\frac{du_{\text{rad}}}{dr} = \frac{h}{\lambda} L(r) \]

Radiative acceleration:

\[ a_{\text{rad}} = \frac{h/\rho \ L(r)}{4\pi r^2 c} \]

Requirement for stable star:

\[ a_{\text{rad}}(R) < g(R) \]

$g$: gravity

\[ g(R) = \frac{GM}{R^2} \]

\[ \Rightarrow \text{upper limit for luminosity} \]

\[ L(R) < \frac{4\pi c GM}{h/\rho} = L_{\text{Edd}} \]

$L_{\text{Edd}}$: Eddington luminosity
Assumption: $\frac{\partial \tau_\nu}{\partial \nu} = 0$

$\tau_\nu = \tau$ independent of $\nu$

Simple but not realistic

Integrating $\int d\nu$ over all equations and quantities: $F = \int F_\nu d\nu$ ....

Consider

$$\mu \frac{dT}{d\tau} = I - S$$

$$\frac{A}{4\pi} \frac{dF}{d\tau} = \gamma - S$$

Radiative equilibrium: $\frac{dF}{d\tau} = 0$

"energy conservation"

$\gamma = S$

$$\mu \frac{dT}{d\tau} = I - \gamma = I - \frac{A}{4\pi} \int I d\nu$$

Eddington's equation for grey transport

no LTE, no isophote for I
Assumption \( \frac{\partial I}{\partial y} = 0 \)

axisymmetry

Power series solution for \( I(\tau, \Theta) \)

\[
I(\tau, \Theta) = I_0(\tau) + I_\alpha(\tau) \cos \Theta
\]

Consequence,

\[
F = \frac{1}{\alpha} \int \cos \Theta d\omega = \frac{I_\alpha}{\pi} \int \cos^2 \Theta d\omega = \frac{I_\alpha}{\pi} \frac{4\pi}{3}
\]

\[
\cos \Theta \frac{dI_0}{d\tau} + \cos^2 \Theta \frac{dI_\alpha}{d\tau} = I_0 + I_\alpha \cos \Theta
\]

\[
-I_0 \int_{\omega} \frac{d\omega}{4\pi} - I_\alpha \int_{\omega} \frac{\cos \Theta d\omega}{4\pi} = 0
\]

\[
\frac{dI_\alpha}{d\tau} = 0 \quad \Rightarrow \quad I_\alpha, F \text{ constant} \quad ; \quad I_\alpha = \frac{3}{4} F
\]

\[
\frac{dI_0}{d\tau} = I_\alpha \quad \Rightarrow \quad I_0 = I_\alpha \tau + \text{constant} = \frac{3}{4} F \tau + \text{constant}
\]

\[
I(\tau, \Theta) = \frac{3}{4} F \tau + \text{constant} + \frac{3}{4} F \cos \Theta
\]

Boundary condition: no incoming radiation at the "surface"

\[
F^-(\tau=0) = 0
\]
\[ f^{-}(t=0) = \frac{1}{u} \int_{\frac{\pi}{2}}^{\pi} d\Theta \int_{-\pi}^{\pi} d\phi \ I(0, \Theta) \cos \Theta \, d\omega \]

\[ = \frac{\text{constant}}{\pi} \int_{-\pi}^{\pi} \cos \Theta \, d\omega + \frac{3}{4\pi} \int_{-\pi}^{\pi} \cos^2 \Theta \, d\omega \]

\[ F^{-}(t=0) = 0 \quad \Rightarrow \quad \text{constant} = F/2 \]

\[ I(t, \Theta) = \frac{F}{2} \left( \lambda + \frac{3}{2} t + \frac{3}{2} \cos \Theta \right) \]

\[ \tau \gg 1 \quad \text{isotropic radiation field} \]

\[ \tau \ll 1 \quad \text{anisotropy large} \]

Application: limb darkening

\[ \frac{I(0, \Theta)}{I(0, 0)} = \frac{2}{5} \left( \lambda + \frac{3}{2} \cos \Theta \right) \]
Assumptions:
1) Isotropy \( I = \frac{y}{\Omega} \)
2) \( I_\nu = B_\nu \); \( I = B = \frac{5}{16} T^4 \)

Anisotropy: temperature cannot be defined.

\[ \Rightarrow \text{ Use } \frac{y}{\Omega} \text{ instead of } I \text{ for definition: } \]

\[ y = \frac{5}{16} T^4 \]

\[ y = \int I \frac{d\omega}{4\pi} = \frac{E}{2} \left( 1 + \frac{3}{2} \tau \right) = \frac{5}{16} T^4 \]

Temperature for \( \tau = 0 \):

\[ \sigma T_\text{eff}^4 (\tau = 0) = \frac{\pi F}{2} \]

Stefan-Boltzmann law, definition of effective temperature:

\[ \sigma T_\text{eff}^4 = \pi F \]

\[ \Rightarrow T^4 (\tau = 0) = \frac{1}{2} T_\text{eff}^4 \]
Temperature stratification of the grey atmosphere

\[ T^4(z) = T(z=0)^4 \left( 1 + \frac{3}{2} z \right) \]
\[ = \frac{1}{2} T_{\text{eff}}^4 \left( 1 + \frac{3}{2} z \right) \]

\[ T = T_{\text{eff}} \text{ for } z = \frac{2}{3} \]

\[ z = \frac{2}{3} : \text{ Photosphere} \]

Stellar interior : \[ z \geq \frac{2}{3} \]
- Consider \( \tau = 2/3 \)
- Radiation field isotropic
- Diffusion approximation for radiation transport

\[
F = - \frac{4 \pi c}{3} \frac{T^3}{\kappa p} \nabla T
\]

\( \kappa \): specific opacity (Romland) \( \kappa = \kappa / \rho \)

\[
L(r) = - \frac{16 \pi c}{3 \kappa p} r^2 T^3 \frac{dT}{dr}
\]

- Outer boundary: Photosphere \( \tau = 2/3 \)
- Boundary condition: Stefan - Boltzmann

\[
L = 4\pi R^2 \frac{2}{3} T_{\text{eff}}^4
\]
Stellar parameters

- Luminosity $L$
  absolute bolometric magnitude $M_B$

- Effective temperature
  colour, e.g. B-V
  spectral type

- Mass

Hertzsprung-Russell diagram
HRD

$L \leftrightarrow T_{\text{eff}}$

Stefan-Boltzmann:

$L = 4\pi R^2 \sigma T^4_{\text{eff}}$

- Main sequence
- Giants
- White dwarfs

HRD of star clusters: Isochrones
Figure 3.2. Hertzsprung-Russell diagram for stars in solar vicinity, showing absolute photographic magnitude versus spectral type. Solid lines are luminosity classes (approximate).

Figure 3.5. Color-magnitude diagram for selected galactic clusters (Population I).
Stellar structure

Definition: Star = Gaseous configuration in hydrostatic equilibrium of pressure gradients and gravitational.

First approximation: Spherical symmetry
No rotation
No magnetic fields
Single objects

Description:

Euler

\( r, t \)

independent variables

Lagrangian

\( r_0, t ; r_0(t) \)

indep. var.

\( r_0 \): initial position

of fixed mass element
Let us consider

\[ M_r = \int_0^r \rho \pi r^2 \, dv \quad M_r \in [0, M] \]

the mass \( M_r \) inside a given radius \( r \) as independent variable rather than \( r \): \( r(M_r) \):

\[ \frac{\partial r}{\partial M_r} = \frac{1}{4\pi \rho r^2} \]

Gravitational Potential \( \phi \):

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) = 4\pi G \rho \]

\[ r^2 \frac{\partial \phi}{\partial r} = G \int_0^r \rho \pi r^2 \, dv = GM_r \]

Gravity \( g \):

\[ g = -\frac{\partial \phi}{\partial r} = -\frac{GM_r}{r^2} \]
Hydrostatic equilibrium

Consider spherical shell

\[ r \quad dr \quad r + dr \]

\[ p(r) \quad p(r + dr) \]

Force exerted on shell:

\[ [p(r + dr) - p(r)] 4\pi r^2 = \frac{dp}{dr} \quad dr \quad 4\pi r^2 \]

Gravitational force on shell,

\[ g \quad dm = g \quad p \quad 4\pi r^2 \quad dr \]

Hydrostatic equilibrium:

\[ \frac{dp}{dr} = g \quad s \]

\[ \frac{1}{s} \quad \frac{dp}{dr} = g = - \frac{GM_r}{r^2} \]

Transformations to \( M_r \):

\[ \frac{dp}{dM_r} = - \frac{GM_r}{4\pi r^4} \]

Boundary condition:

\[ p(M_r = M) = 0 \]
Violation of equilibrium leads to acceleration.

Newton's law:

\[ \frac{d^2 r}{dt^2} = - \frac{1}{s} \frac{\partial P}{\partial s} - \frac{GM_r}{r^2} \]

Transformation to \( M_r \):

\[ \frac{\partial P}{\partial M_r} = - \frac{GM_r}{4\pi r^4} - \frac{1}{4\pi r^2} \frac{d^2 r}{dt^2} \]
Asymptotic Pressure given by equation of state of ideal gas:

\[ p = \frac{\mathcal{R}}{\mu} sT \; ; \; c_v = \frac{3}{2} \frac{\mathcal{R}}{\mu} \]

Thermal energy

\[ E_T = \int c_v T \, \, dM_v = \int \frac{3}{2} \frac{\mathcal{R}}{\mu} T \, \, dM_v \]

Potential energy of mass element at \( r \):

\[ dE_G = dM_v \int \frac{GM_v}{v^2} \, dv = - \frac{GM_v}{r} \, dM_v \]

Total potential energy

\[ E_G = -\int \frac{GM_v}{r} \, dM_v \]

Consider Mechanical equilibrium

\[ \frac{dp}{dM_v} = -\frac{GM_v}{\mu r^4} \times 4\pi r^3 dM_v ; \int \]
\[
\int_0^M 4\pi r^3 \frac{dp}{dM_r} \, dM_r = - \int_0^M \frac{GM_r}{r} \, dM_r = E_g
\]

\[
= \left[ 4\pi r^3 p \right]_0^M - \int_0^M 12\pi r^2 \, \frac{dv}{dM_r} \, \rho \, dM_r
\]

\[
= -3 \int_0^M \frac{\rho R}{\mu} \, T \, dM_r = -3E_I = -2E_T
\]

**Virial Theorem:**

\[E_g = -2E_T \uparrow\]

depends on equation of state

**Total energy:**

\[E_{\text{tot}} = E_g + E_T = -E_T < 0 \uparrow\]

(gravitationally) bound system

Virial theorem ≠ Energy conservation!!

Virial theorem consequence of hydrostatic equilibrium
Contraction of a star:

$$-\delta E_g > 0 \implies \delta E_T = -\frac{1}{2} \delta E_g > 0$$

Star becomes "hotter"

$$\delta E_{tot} = -\delta E_T = \frac{1}{2} \delta E_g < 0$$

Total energy decreases due to energy loss by radiation

$$\delta E_{tot} < 0; \delta E_T > 0: \text{negative specific heat}$$

Time scale for thermal evolution:

Klein--Kuhn--Kollwitz - time scale $T_{KH}$

$$|E_T| \sim |E_g| \sim \frac{GM^2}{R}$$

$$T_{KH} \sim \frac{|E_T|}{L} \sim \frac{|E_g|}{L} \sim \frac{GM^2}{RL}$$

Sum:

$L_0 = 4 \cdot 10^{33}$ erg/sec

$M_0 = 2 \cdot 10^{33}$ g

$R_0 = 7 \cdot 10^{14}$ cm

$T_{KH} \sim 3 \cdot 10^7$ years $< T_{Earth} \approx 4 \cdot 10^9$ years

$\implies$ Gravitational energy is not the source of stellar radiation.
Energy conservation

$L_r$ : energy/mass flowing through spherical shell at radius $r$

Boundary conditions:

$L_r (M_r = 0) = 0$ ; $L_r (M_r = M) = L$  
(observable luminosity)

$L_r (M_r)$ non-monotonic:

\[\begin{align*}
L_r & \uparrow \\
M_r & \\
\end{align*}\]

Change of $L_r$ per mass shrinkage:

\[
\frac{\partial L}{\partial M_r} = E_N - E_r + E_g
\]

nuclear energy \hspace{1cm} neutrino losses \hspace{1cm} heat content

\[\begin{align*}
E_g &= - \frac{dQ}{dT} = - T \frac{dS}{dT} \\
\text{thermal equilibrium} \\
\text{adiabatic changes of state} \quad \Rightarrow \quad \boxed{E_g = 0}
\end{align*}\]
\[ \varepsilon_g = -T \frac{\partial s}{\partial t} \]

\[ S = S(p, T) = S(p, T) = \ldots \]

From \( S = S(p, T) \):

\[ \varepsilon_g = -c_p \frac{dT}{dt} - T \left. \frac{\partial S}{\partial p} \right|_T \frac{dp}{dt} \]

1st law of thermodynamics

\[ dQ = du - p\frac{\partial s}{\partial p} dp \]

\[ u = u(p, T) \quad ; \quad s = s(p, T) \quad ; \quad S = S(p, T) \]

\[ dQ = \left( \frac{\partial u}{\partial T} \right)_p - p\frac{\partial s}{\partial p} \frac{\partial p}{\partial T} dp \]

\[ + \left( \frac{\partial u}{\partial p} \right)_T - p\frac{\partial s}{\partial T} \frac{\partial s}{\partial p} \right|_T dp \]

\[ = TdS = T \left. \frac{\partial S}{\partial T} \right|_p \frac{dT}{dt} + T \left. \frac{\partial S}{\partial p} \right|_T \frac{dp}{dt} \]

\[ - \\frac{\partial S}{\partial T} \left|_p = \frac{\partial u}{\partial T} \right|_p - p\frac{\partial s}{\partial p} \frac{\partial p}{\partial T} \left|_p \right. \quad (1) \]

\[ + \left. \frac{\partial u}{\partial p} \right|_T - p\frac{\partial s}{\partial T} \frac{\partial s}{\partial p} \right|_T \left( 2 \right) \]

\[ \left. \frac{\partial \theta}{\partial p} \right|_T \left( 1 \right) \quad - \frac{\partial \theta}{\partial T} \left|_p \right. \left( 2 \right) : \]

\[ - \left. \frac{\partial \theta}{\partial p} \right|_T = - \left. \frac{\partial \theta}{\partial T} \right|_p \left( \frac{4}{3} - p \frac{2}{3} \frac{\partial \theta}{\partial p} \right) \]

\[ + \left. \frac{\partial \theta}{\partial T} \right|_T \left( -p \frac{2}{3} \frac{\partial \theta}{\partial p} \right) = - \left. \frac{\partial \theta}{\partial T} \right|_p \frac{4}{3} \frac{\partial \theta}{\partial p} \]

\[ \left|_T \right. \]
\[ \dot{E}_g = -c_P \frac{dT}{dt} - \frac{T}{\rho^2} \frac{\partial \rho}{\partial t} |_\rho \frac{d\rho}{dt} \]

Definition

\[ s = - \frac{\partial \log \rho}{\partial \log T} |_\rho = - \frac{T}{S} \frac{\partial S}{\partial T} |_\rho \]

\[ \dot{E}_g = -c_P \frac{dT}{dt} + \frac{s}{S} \frac{d\rho}{dt} \]
1) Mechanical (dynamical, free-fall) timescale

\[ \frac{d^2 r}{dt^2} \sim \frac{GM}{r^2} \; ; \; \frac{1}{R} \frac{dP}{dr} \]

\[ \frac{R}{T_{\text{Dyn}}} \sim \frac{GM}{R^2} \]

\[ T_{\text{Dyn}} \sim \left( \frac{R^3}{GM} \right)^{1/2} \sim (G \langle \rho \rangle)^{-1/2} \]

Examples:

- Sun: \( T_{\text{Dyn}} \sim 25 \) min
- Giant: \( R \sim 10^{-2} R_\odot \) \( \sim 20 \) d
- White dwarf: \( R \sim 10^{-2} R_\odot \) \( \sim 1.5 \) sec
- Neutron star: \( R \sim 10^{-5} R_\odot \) \( \sim 10^{-4} \) sec
2) Thermal (KH) timescale

\[ \tau_{KH} = \frac{GM^2}{RL} \]

3) Nuclear timescale

\[ \tau_{\text{nuc}} = \frac{E_{\text{nuc}}}{L} = \left(\frac{\Delta M}{M}\right)_{\text{nuc}} \frac{Mc^2}{L} \]

\[ \left(\frac{\Delta M}{M}\right)_{\text{nuc}} \approx 7 \times 10^{-3} \text{ at maximum} \]

Sun: \[ \tau_{\text{nuc}} \sim 10^{20} \text{ years} \]

Relations

\[ \tau_{\text{nuc}} \gg \tau_{KH} \gg \tau_{\text{dyn}} \]

\[ \Rightarrow \] Hydrostatic equilibrium satisfied for stellar evolution.

But: dynamical phases at certain stages of evolution.
Energy transport in stars

Transport by diffusion of radiation:

\[ L_r(v) = - \frac{16 \pi \sigma c}{3 \bar{\kappa} \rho} r^2 T^3 \frac{dT}{dr} \]

\( \bar{\kappa} \): specific opacity (Rosseland mean)

Conductive transport

not relevant in ordinary stars

photon mean \( \rightarrow \) particle (electrons) mean free path

\( \rightarrow \) free path

degenerate stars: situation may be reversed

Description of conduction as diffusion process:

\[ L_{\text{conductive}} = - D_{\text{conductive}} \frac{dT}{dr} \]

For convenience, define conductive opacity \( \kappa_{\text{read}} \) such that radiation diffusion equation remains valid with total opacity \( \kappa_{\text{tot}} \):

\[ \frac{\Lambda}{\kappa_{\text{tot}}} = \frac{\Lambda}{\kappa} + \frac{\Lambda}{\kappa_{\text{read}}} \]
\[ \frac{\partial T}{\partial M_r} = -\frac{3}{64 \pi^2 ac} \frac{R}{r^4 T^3} L_r \]

\[ =: \nabla \frac{I}{P} \frac{\partial P}{\partial M_r} \]

Definition of \( \nabla \)

\[ \nabla = \nabla_{\text{rad}} = \frac{3}{16 \pi \gamma_a c} \frac{R L_r P}{T^4 G M_r} \]

General treatment of energy transport by

\[ \frac{\partial T}{\partial M_r} = \nabla \frac{I}{P} \frac{\partial P}{\partial M_r} \]

Details of transport determine \( \nabla \)

e.g., radiative transport \( \nabla = \nabla_{\text{rad}} \)

conductive transport \( \nabla = \nabla_{\text{rad}} \)

with \( \frac{1}{R_{\text{tot}}} = \frac{1}{R} + \frac{1}{R_{\text{cond}}} \)

Radiatively determined temperature gradient too steep \( \rightarrow \) instability of stratification

\( \rightarrow \) "Convection"

\( \rightarrow \) energy transport by convection
Stability of Stratification
Criteria for the onset of convection

\[ F = -g \Delta \rho \]
\[ \Delta \rho = \rho_{\text{Element}} - \rho_{\text{surroundings}} = \Delta \rho_E - \Delta \rho_S \]
\[ \Delta \rho_S = \frac{d\rho}{d\xi} \Delta \xi \]

Assumptions:

1) Pressure equilibrium
   \[ \Delta P_{\text{Element}} = \Delta P_{\text{surroundings}} \]
   \[ P_{\text{Element}} = P_{\text{surroundings}} \]

2) Adiabatic change of state for element, displacement sufficiently fast: \[ P_E = P_E(S_E) \]
   \[ \Delta P_{\text{Element}} = \frac{dP}{dS} \bigg|_{\text{ad}} \Delta S_{\text{Element}} \]
   \[ = \gamma_{\text{ad}} P/S \]
   \[ \gamma_{\text{ad}} = \frac{\partial \ln P}{\partial \ln S} \quad S: \text{entropy} \]
\[ \Delta p_E = \Delta p_{\mathcal{E}} = \Delta p_s = \frac{d\rho}{dr} \Delta r \]

\[ F = -g \left( \frac{\rho}{\gamma \Delta p} \frac{d\rho}{dr} - \frac{d\rho}{dr} \right) \Delta r \]

\[ F = g \Delta r \left( \frac{d\rho}{dr} - \frac{\rho}{\gamma \Delta p} \frac{d\rho}{dr} \right) \]

\begin{align*}
\leq 0 & : \text{stability} \\
> 0 & : \text{instability}
\end{align*}

One formulation of criterion for convective instability

Relation to Rayleigh - Taylor instability

Assumption: Fluid incompressible

\[ \Delta p_E = 0 \]

\[ \Rightarrow \Delta p = -\Delta p_s \]

\[ F = g \Delta r \frac{d\rho}{dr} \]

RT-instability and convective instability rely on the same physics with different properties of matter

\[ \Rightarrow \text{criteria must not be used simultaneously!} \]
General stability criterion:

\[ Q := \frac{d\varepsilon}{dv} - \frac{\psi}{\gamma_{av} p} \frac{dp}{dv} \leq 0 \quad \text{stability} \quad \text{instability} \]

Equation of state

\[ \psi = \psi(p, T, \mu) \quad \mu: \text{molecular weight, chemical composition} \]

Differential form:

\[ \frac{1}{s} \frac{ds}{dv} = \alpha \frac{1}{p} \frac{dp}{dv} - \delta \frac{1}{T} \frac{dT}{dv} + \gamma \frac{1}{\mu} \frac{d\mu}{dv} \]

\[ \alpha = \left. \frac{\partial \log p}{\partial \log p} \right|_{T, \mu} \quad \delta = - \left. \frac{\partial \log T}{\partial \log T} \right|_{p, \mu} \]

\[ \gamma = \left. \frac{\partial \log \mu}{\partial \log \mu} \right|_{T, \mu} \]

\[ Q = s \frac{1}{p} \frac{dp}{dv} \delta \left( \frac{\alpha}{\delta} (\alpha - \frac{A}{\gamma_{av}}) - \frac{\partial \log T}{\partial \log p} + \frac{\gamma}{\delta} \frac{d\log \mu}{d\log p} \right) \]

\[ \Delta \leq \frac{A}{\delta} (\alpha - \frac{A}{\gamma_{av}}) + \frac{\gamma}{\delta} \frac{d\mu}{dp} \quad \text{stability} \quad \text{instability} \]

Radiative transport:

\[ \Delta = \Delta_{\text{rad}} \]
\[ \frac{\Delta}{\Delta_{ad}} = \frac{\partial \log s}{\partial \log p} \frac{\partial \log p}{\partial \log s} \]

\[ \Delta_{ad} = \frac{\partial \log s}{\partial \log p} \]

Entirely determined by equation of state

\[ \frac{\Delta}{\Delta_{ad}} = \alpha - \frac{\partial \log p}{\partial \log s} = \frac{\partial \log p}{\partial \log s} - \frac{\partial \log s}{\partial \log p} \]

Stability criterion:

\[ \nabla_{ad} \leq \nabla_{ad} + \frac{\gamma}{\delta} \nabla_{\mu} \]

Stability

\[ \nabla_{\mu} = 0 \quad \text{Ledoux - criterion} \]

\[ \nabla_{\mu} = 0 \quad \text{Schwarzschild - criterion} \]

Remark: Most general form of criteria may be written as

\[ \frac{ds}{dr} \geq 0 \quad \text{stability} \]

\[ \frac{ds}{dr} < 0 \quad \text{instability} \]

Physical origin: buoyancy
consistent theory of convection!!

Problem: calculate $\nabla$

Limiting case: convection effective (stellar core),
Consequence: reduces $\nabla$
Marginally stable state:
$\nabla = \nabla_{ad}$
Convection adiabatic

General case: convection insufficient
non-adiabatic

$\nabla_{ad} < \nabla < \nabla_{rad}$

"Theory" for non-adiabatic convection:
Mixing-length theory MLT (Prandtl, Bohm-Vitense)

Parameters of MLT: Mixing length

$$l = \alpha \left( \frac{1}{H_p} \right)^{\frac{2}{\gamma}}$$

$H_p$ = pressure scale height

$\alpha \sim O(1)$
$\alpha = 1 \ldots 2$

Result of MLT: $\nabla_{MLT}$: $\nabla_{ad} < \nabla_{MLT} < \nabla_{rad}$
In convective region total flux consists of radiative and convective part

\[ F_{\text{tot}} = F_{\text{rad}} + F_{\text{con}} \]

Definition of \( F_{\text{rad}} \):

\[ F = \frac{4ac}{3} \frac{T^4}{KgHo} \nabla_{\text{rad}} \tag{1} \]

Definition of \( \nabla \):

\[ F_{\text{rad}} = \frac{4ac}{3} \frac{T^4}{KgHo} \nabla \tag{2} \]

Energy flux by moving (convective) mass element

\[ F_{\text{con}}' = c_p \Delta T \rho \bar{u} \]

\( \Delta p = 0 \): pressure equilibrium

\( \Delta T = T_{\text{element}} - T_{\text{surrounding}} \)

Total convective flux: suitable average

\[ F_{\text{con}} = \rho \bar{u} c_p \Delta T \quad \bar{u}, \Delta T = ? \]
Parameters of MRT: Mixing length $l$ corresponds to mean free path of convective element

$r_0: \Delta T(r_0) = 0; \Delta p(r_0) = 0$

Taylor expansion around $r_0$:

\[
\Delta T = \Delta T(r_0) + \frac{d \Delta T}{dr} (r - r_0)
\]

\[
= \left( \frac{d T_{\text{mean}}}{dr} - \frac{d T_{\text{sum}}}{dr} \right) (r - r_0)
\]

\[
= -\frac{d}{H_p} \left( \frac{\partial \log T_e}{\partial \log p} - \frac{\partial \log T_s}{\partial \log p} \right) (r - r_0)
\]

\[
\Delta p = \Delta p(r_0) + \frac{d \Delta p}{dr} (r - r_0)
\]

\[
= \left( \frac{d \rho e}{dr} - \frac{d \rho s}{dr} \right) (r - r_0)
\]

\[
= -\frac{\bar{p}}{H_p} \left( \frac{\partial \log \rho e}{\partial \log p} - \frac{\partial \log \rho s}{\partial \log p} \right) (r - r_0)
\]

Equation of state:

\[
\Delta p = -\frac{\bar{p}}{H_p} \left( \alpha - \delta \nabla e - \alpha + \delta \nabla \right) (r - r_0)
\]

Identify $\bar{p} = \bar{T}; \bar{p} = \bar{s}$
\[ \Delta T = - \frac{I}{H_p} (\nabla_e - \nabla) (r-r_0) \]

\[ \Delta \rho = \frac{g \delta}{H_p} (\nabla_e - \nabla) (r-r_0) \]

Change of potential energy for the rising element:

\[ \Delta E = \int_{r}^{r_0} \rho \Delta \rho \, dr = \frac{g \delta}{H_p} \int_{r}^{r_0} (\nabla_e - \nabla) (r-r_0) \, dr \]

\[ \Delta E = \frac{g \delta}{H_p} (\nabla_e - \nabla) \left( \frac{r-r_0}{2} \right)^2 \]

Assumptions:
- Work needed for expansion \( \sim \frac{1}{2} \Delta E \)
- Kinetic energy \( \sim \frac{1}{2} \Delta E \)

\[ \Delta E_{\text{kin}} = \frac{1}{2} g \delta v^2 = \frac{1}{2} \Delta E = \frac{g \delta}{2H_p} (\nabla_e - \nabla) \left( \frac{r-r_0}{2} \right)^2 \]

Velocity of convective element:

\[ v = \left[ \frac{g \delta}{2H_p} (\nabla_e - \nabla) \right]^{1/2} (r-r_0) \]

Average of \( \Delta T \) and \( v \) over mixing length \( l \):

\[ \bar{v} = \frac{1}{l} \int_{r_0-l}^{r_0} |v| \, dr \quad ; \quad \bar{\Delta T} = \frac{1}{l} \int_{r_0-l}^{r_0} |\Delta T| \, dr \]
\[ \Delta T = \frac{T}{2H_p} \left( \nabla - \nabla_e \right) l \]
\[ \overline{V}^2 = \frac{g_0}{2H_p} \left( \nabla - \nabla_e \right) \frac{l^2}{4} \quad (3) \]
\[ F_{cm} = g_0 \overline{V} C_p \Delta T \]
\[ F_{cm} = g C_p T \left( g_0 \right)^{1/2} \frac{l^2}{4 \sqrt{2} H_p^{3/2}} \left( \nabla - \nabla_e \right)^{3/2} \quad (4) \]

\[ \Rightarrow 4 \text{ equations for unknowns} \]
\[ F_{cm}, \text{ Fred, } \nabla_e, \nabla, \overline{V} \]

5th equation of MLT: Consider energy loss of convective element driven motion.

Radiative flux:
\[ F = - \frac{4ac}{3} \frac{T^3}{K_p} \frac{dT}{dr} \]
\[ \frac{dT}{dr} \sim \frac{\Delta T}{l/2} \quad \text{dimension of convective element } \sim l \]

Energy loss per mass and time
\[ \dot{Q} = F \frac{S}{\rho V} \]
\[ \dot{Q} = - \frac{8ac}{3} \frac{T^4}{K_p^2} \frac{\nabla - \nabla_e}{l H_p} \frac{S}{V} \left( r - r_o \right) \]
Total energy loss:

$$\Delta Q = \int \dot{Q} \, dr = \int \frac{\dot{Q}}{v} \, dr = \frac{\dot{Q}}{v} (r - r_0)$$

[\text{Q/v independent of r}]

Average over mixing length:

$$\bar{\Delta Q} = \frac{\dot{Q}}{v} \frac{L}{2} = - \frac{2ac}{3} \frac{T^4}{k_s^2} \frac{L}{H_p} \frac{\nabla - \nabla e}{V}$$

Energy loss in terms of thermodynamic quantities:

$$\Delta Q = T \Delta S \text{elum} = T \left( \frac{\partial S}{\partial p} \bigg|_T \Delta p_e + \frac{\partial S}{\partial T} \bigg|_p \Delta T_e \right)$$

$$= \frac{T^2}{P} \Delta p_e \left( \frac{P}{T} \frac{\partial S}{\partial P} \bigg|_T \frac{\partial S}{\partial T} \bigg|_P + \frac{P}{T} \frac{\partial T_e}{\partial p} \frac{\partial T_e}{\partial p} \bigg|_T \right)$$

$$= \frac{T^2}{P} \Delta p_e \left( \frac{c_p}{T} - \nabla \text{ad} \right)$$

$$\Delta T_e = \frac{\partial T_e}{\partial p} \Delta p_e$$

$$\frac{\partial S}{\partial P} \bigg|_T \frac{\partial S}{\partial T} \bigg|_P = - \frac{\partial (T,S)}{\partial (P,T)} \frac{\partial (T,P)}{\partial (S,P)} = - \frac{\partial (T,S)}{\partial (P,S)}$$

$$= - \frac{\partial T}{\partial P} \bigg|_S$$

$$\frac{\partial (u,y)}{\partial (x,y)} : = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \quad \frac{\partial (u,y)}{\partial (x,y)} = \frac{\partial u}{\partial x} \bigg|_y$$
Properties: anti-symmetric
in u, v and x, y

\[ \frac{\partial u}{\partial t} = \frac{\partial (u,v)}{\partial (t,x,y)} = \frac{\partial (u,v)}{\partial (x,y)} \]

\[ \Delta Q = c_p T (\nabla e - \nabla ad) \frac{\Delta P e}{P} \]

\[ \Delta P e = \Delta P s = P_s(r) - P_s(r_0) = \frac{2P}{3r} (r - r_0) \]

\[ \Delta Q = -c_p T (\nabla e - \nabla ad) \frac{4}{4P} (r - r_0) \]

Average over mixing length:

\[ \bar{\Delta Q} = -c_p T (\nabla e - \nabla ad) \frac{4}{4P} \frac{l}{2} \]

\[ \frac{\nabla e - \nabla ad}{\nabla - \nabla e} = \frac{4ac}{3} \frac{T^3}{K_p^2 c_p V} \frac{S}{V} \]

S/V with dimension of convective element l:

sphere: \[ S/V = \frac{3}{l} \] (minimum)
cube: \[ S/V = \frac{6}{l} \]

Adopt \[ S/V = \frac{9/2}{l} \] for standard MLT
Equations of Mixing Length Theory

\[ F_{\text{tot}} = F_{\text{rad}} + F_{\text{cm}} = \frac{4ac}{3} \frac{T^4}{\kappa g H_p} \nabla_{\text{rad}} \]  
\[ F_{\text{rad}} = \frac{4ac}{3} \frac{T^4}{\kappa g H_p} \nabla \]  
\[ F_{\text{cm}} = \rho C_p T (g \delta)^{1/2} \frac{l^2}{4 \sqrt{2} H_p} (\nabla - \nabla_e)^{3/2} \]  
\[ \overline{\nabla^2} = g \delta (\nabla - \nabla_e) \frac{l^2}{8 H_p} \]  
\[ \frac{\nabla_e - \nabla_{\text{rad}}}{\nabla - \nabla_e} = \frac{6ac T^3}{\kappa g^2 c_p \overline{\nabla^2}} \]

Free parameters: Mixing length \( l \)

Relation to typical length scale:

- Premixed Scale Length
- Density Scale Height

usually:

\[ l = \alpha H_p \]  
\[ \alpha = o(1) = 1 \ldots 2 \]

Solution of MLT equations with prescribed

\[ F_{\text{tot}}, T, P, g, R, H_p, l, C_p, \delta, S, \nabla_{\text{rad}}, \nabla_{\text{cm}} \]

unknown:

\[ F_{\text{rad}}, F_{\text{cm}}, \overline{\nabla}, \nabla, \nabla_e \]
Procedure for solution

Add (2) and (3) ⇒ 1. equation in $D, D_e$ (A)
Insert (4) into (5) ⇒ 2. equation in $D, D_e$ (B)

$A, B$ represent two algebraic equations for $D, D_e$
lead to cubic equation with a single real solution, analytically representable

Determine $D, D_e \Rightarrow \nabla (4), \text{ From } (3), \text{ From } (2)$

Limiting cases:

Convection efficient $gcp$ large $\Rightarrow D \rightarrow D_{ad}$ stellar core

Convection inefficient $gcp$ small $\Rightarrow D \rightarrow D_{rad}$ stellar envelope

MLT usually not used in stellar cores

Convection overshooting:

Convection elements penetrate into radiative, convection stable region
measured in terms of $H_p$, value ?
Mixture of nuclei with mass \( m_i \), number density \( n_i \), total mass density \( \rho \)

**Mass fraction:**

\[
X_i = \frac{m_i \cdot n_i}{\rho}
\]

\[
\sum X_i = 1
\]

**Hydrogen mass fraction:** \( X_H = X \)

**Helium:** \( X_{\text{He}} = Y \)

**Remaining:** \( \sum_{i \neq H, \text{He}} X_i = Z \)

**Remaining nuclei:** "Metals"; "Heavy elements"

\[
Z = 1 - X - Y
\]

**Typical values:**

\[
X = 0.6 \ldots 0.7
\]

\[
Y = 0.36 \ldots 0.3
\]

\[
Z = 0.04 \ldots 0.001
\]

**Chemical composition fraction of mass coordinate**

\[
X_i = X_i (M_v, t)
\]

↑ evolution
Variation of composition with time

\[
\frac{\partial x_i}{\partial t} = \frac{m_i}{S} \left[ \frac{\Xi}{\beta} \gamma_{ji} - \frac{\Xi}{\beta} \gamma_{ik} \right]
\]

\(\gamma_{ji} : \text{reaction rate} \quad j \to i\)

\(\gamma_{ik} : \quad i \to k\)

\(\beta : \text{reaction per volume and time (T, e)}\)

(nuclear physics)

Change of composition associated with (nuclear) energy generation

reaction \(j \to i\): energy release \(Q_{ji}\)

energy generation rate:

\[
\dot{E}_{ji} = \frac{Q_{ji} \gamma_{ji}}{S}
\]

Total energy generation rate

\[
\dot{E}(S, T) = \sum \dot{E}_{ji}
\]

Convection requires: instantaneous mixing

composition constant

\[
\frac{\partial x_i}{\partial M_T} \bigg|_{\text{conv}} = 0
\]
Differential equations of stellar structure and evolution

\[ \frac{\partial \nu}{\partial M_r} = \frac{\Lambda}{4\pi \nu^2 r} \]  \hspace{1cm} (1)

\[ \frac{\partial \rho}{\partial M_r} = -\frac{GM_r}{4\pi r^4} - \frac{\Lambda}{4\pi r^2} \frac{\partial^2 \nu}{\partial \nu^2} \]  \hspace{1cm} (2)

\[ \frac{\partial L_r}{\partial M_r} = \varepsilon_{\text{me}} - \varepsilon_r - \xi_m \frac{\partial T}{\partial \nu} + \frac{\xi}{s} \frac{\partial \rho}{\partial \nu} \]  \hspace{1cm} (3)

\[ \frac{\partial T}{\partial M_r} = -\frac{GM_r T}{4\pi r^4 \rho} \]  \hspace{1cm} (4)

\[ \frac{\partial X_i}{\partial \nu} = \frac{m_i}{s} \left[ \frac{\Xi_i}{\delta} r_{ji} - \frac{\Xi_i}{\delta} r_{ji} \right] \]  \hspace{1cm} (5)

with

\[ \nabla_{\text{ad}} < \nabla = \nabla_{\text{MLT}} < \nabla_{\text{rad}} \]

in convective regions

\[ \nabla = \nabla_{\text{rad}} \]

in radiative regions

\[ \varepsilon_{\text{me}} = \varepsilon_{\text{me}} (p, T, X_i) \]

\[ \varepsilon_r = \varepsilon_r (p, T, X_i) \]

\[ m_i = m_i (p, T, X_i) \]

\[ K = K (p, T, X_i) \]

\[ r_{ji} = r_{ji} (p, T, X_i) \]  nuclear physics

\[ \Xi_i = \Xi_i (p, T, X_i) \]  atomic physics
Closure of system by prescription of equation of state \((EoS)\):

\[
\rho = \rho (p, T, x_i)
\]

\(EoS\) implies:

\[
C_p = C_p (p, T, x_i) \quad \text{and} \quad \gamma = \gamma (p, T, x_i) \quad \text{and} \quad V_{ad} = V_{ad} (p, T, x_i)
\]

\(T\) time scales

\[
T_{nuc} >> T_{KH} >> T_{dyn}
\]

Hydrodynamic equilibrium:

\[
\frac{\partial^2 \rho}{\partial t^2} = 0 \quad \text{in momentum equation}
\]

Thermal equilibrium:

\[
\frac{\partial T}{\partial t} = 0 \quad \text{and} \quad \frac{\partial p}{\partial t} = 0 \quad \text{in energy equation}
\]
Boundary Conditions

Stellar evolution: Mixed initial/boundary value problem

Central boundary conditions:
\[
\tau(M_r=0) = 0
\]
\[
L_r(M_r=0) = 0
\]

Surface boundary conditions:
\[
\rho(M_r=M) = 0 \quad \text{"zero condition"}
\]
\[
T(M_r=M) = 0 \quad \text{not realistic}
\]

Surface of the star: Photosphere \( r = R_{\text{eff}} = 2/3 \)
Stellar - Boltzmann:
\[
T^4(M_r=M) = T_{\text{eff}} = \frac{L_r(M_r=M)}{4\pi G \sigma T^2(M_r=M)}
\]

For second condition consider hydrostatic atmosphere:
\[
\frac{\partial P}{\partial r} = \frac{g}{\sigma} \quad \text{integration from photosphere to } \infty
\]
\[
-P_{\text{eff}} = P(M_r=M) = \int_{R_{\text{eff}}}^{\infty} \frac{g}{\sigma} \, dr = \frac{1}{g} \int_{R_{\text{eff}}}^{\infty} \rho \, dv
\]
\[
V(M_r=M) \quad r(M_r=M)
\]
\[ dT = -k \rho \, dv \]

Integration from infinity to photosphere
\[ r(M_r = M) \]

\[ T_{\text{eff}} = \frac{2}{3} = - \int_{0}^{\infty} k \rho \, dv = - \bar{k} \int_{0}^{\infty} \rho \, dv \]

\[ \frac{2}{3} = \bar{k} \int_{r(M_r = M)} \rho \, dv \]

\[ \frac{2}{3} \frac{\lambda}{\bar{k}} = - \frac{P_{\text{eff}}}{\bar{g}} \]

Approximation:
\[ \bar{R} = R_{\text{eff}} = R(M_r = M) \]

\[ \bar{g} = g_{\text{eff}} = - \frac{GM}{r^2(M_r = M)} \]

Boundary condition:

\[ P_{\text{eff}} = \frac{2}{3} \frac{GM}{r^2(M_r = M)} \frac{\lambda}{R_{\text{eff}}} \]

Qualities of approximations?

Results are not very similar to precise form of second order BC.
Restricted Problem:
Envelope solutions

No nuclear processes, constant (given)
chemical composition $\Rightarrow$ $L = \text{constant}$

Hydrostatic and thermal equilibrium
Prescribe $L$, $T_{eff}$, $M$ (position on HRD, estimate mass)

Problem reduces to initial value problem
for mass conservation
momentum conservation
energy transport

Surface flux inward from the surface
with initial conditions

$M_0 = M$ :  $T = T_{eff}$

$R = \sqrt{M_0/M}$
from Stefan–Boltzmann

$P = P_{eff}$ 2nd order BC
Numerical Solution

Hydrostatic equilibrium assumed

$T_{\text{Evolution}} > T_{\text{KH}}$: (thermal equilibrium)

Variations of components decoupled from thin independent part

\[
\text{Solve (1)-(4) with } X_i \text{ fixed } \rightarrow \text{ new P, T, ...} \\
\downarrow
\]

\[
\text{Solve (5) with P, T, ... fixed } \rightarrow \text{ new } X_i
\]

$T_{\text{Evolution}} \approx T_{\text{KH}}$

Treatment of (1)-(4)/(5) as before but

\[
\text{Solve (1)-(4) with } E_g \text{ fixed } \rightarrow \text{ new P, T, ...} \\
\downarrow
\]

\[
\text{calculate new } E_g \text{ with previous and present P, T, ...}
\]

Boundary value problem (1)-(4):

Discretisation

Solution of difference equation by Newton method

Matrix inversion (block structure)

Block elimination procedure by Henegar
Consider mixture of various nuclei, atoms, (partially) ionized atoms, electrons and radiation.

Ions, atoms and electrons (but: degeneracy) satisfy condition of ideal gas:

\[
P_{\text{ion}} = \frac{\Sigma m_e}{k} kT
\]

\[
P_e = m_e kT
\]

\[
P_{\text{gas}} = P_{\text{ion}} + P_e
\]

For practical purposes desired

\[
P_{\text{gas}} = \frac{R}{\mu} \Sigma T
\]

\[\mu: \text{mean molecular weight}\]

\[
\Sigma = \frac{\Sigma m_e \mu_e m_p + n_e m_e}{k}
\]

\[
P_{\text{gas}} = kT \left( \frac{\Sigma m_e}{k} + n_e \right)
\]

\[
= \frac{kT}{\mu} \left( \Sigma m_e \mu_e + n_e m_e m_e / m_p \right)
\]

\[
\mu = \frac{\Sigma m_e \mu_e + (n_e m_e m_e / m_p)}{\Sigma m_e + n_e}
\]

\[\mu \text{ depends on degree of ionization}\]
Complete Ionization (stellar core)

\[ n_e = \sum \frac{m_k}{\mu_k} z_k \]

\[ x_k = \frac{m_k/\mu_k m_p}{\xi} \]

\[ \sum x_k = 1 \quad ; \quad m_e/m_p \ll 1 \]

\[ \mu = \frac{\sum \frac{m_k/\mu_k m_p}{\xi}}{\sum (1 + 2z_k) \frac{1}{\mu_k} \frac{m_k/\mu_k m_p}{\xi}} \]

\[ \mu = \frac{1}{\sum \frac{1 + 2z_k}{\mu_k} x_k} \]

"Heavy elements": \( z_k \gg \frac{m_k}{\xi} \)

\[ \mu = \frac{1}{2x + \frac{3}{4}y + \frac{1}{2}z} \]

Partial ionization (stellar envelope)

Determine number density of atoms in their various states, various ions correspondingly and we according to thermodynamic equilibrium.
Consider atom in state $i$.

$n_i$: number density; $g_i$: statistical weight

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} \exp\left(-\frac{(E_j - E_i)}{kT}\right)$$

Boltzmann formula

$n_r$: number density of $r$th emission or in ground state

$$\frac{n_{r+1} n_e}{n_r} = \frac{g_{r+1}}{g_r} \frac{2}{(2\pi m_e kT)^{3/2}} \frac{(2\pi m_e kT)^{3/2}}{h^3} \exp\left(-\frac{\chi_r}{kT}\right)$$

$\chi_r$: rotational energy

Saha equation (law of mass action)

Add up all contributions:

- change neutrality
- mass conservation

$$\Rightarrow \mu(p, T)$$

Radiation pressure

$$P_{rad} = \frac{a}{3} T^4$$

Total pressure

$$P = P_{gas} + P_{rad} = \frac{\rho}{\mu} sT + \frac{a}{3} T^4$$
Specific internal energy:

\[ U = \frac{3}{2} \varepsilon \frac{\hbar^2}{m} T + \frac{\hbar^4}{\mathcal{E}^4} + \text{constant} \]

\( \varepsilon, \mathcal{E}, \mathcal{U} \) are deduced either from or by means of standard thermodynamic relations.

Electron pressure at low temperature

Electrons obey Fermi-Dirac statistics.

Probability to find electron with momentum \( \mathcal{E} \) in interval \( d\mathcal{E} \)

\[ dN_p = \frac{g \nu p^2 dp}{2\pi \hbar^2 \left[ \exp \left( \frac{(\mathcal{E} - \mu)/kT}{\hbar} \right) + 1 \right]} \]

- \( g = 2 \) statistical weight
- \( \nu \): volume
- \( \mu \): chemical potential
- \( \mathcal{E}(p) \): energy

Number density

\[ \frac{N_e}{V} = \int_0^\infty \frac{dN_p}{V} \]

Energy density

\[ \frac{E}{V} = \int_0^\infty \mathcal{E}(p) \frac{dN_p}{V} \]
Thermodynamic potential $\Omega$

(see Landau - Lifschitz)

$$\Omega = - \int_{0}^{\infty} \frac{g P^2 dp}{2 \pi^{\frac{3}{2}} \hbar^3} \log [\exp \{(\mu - \epsilon)/kT\} + 1]$$

General property:

$$\Omega = -PV = - PeV$$

For Fermi- Dirac - statistics:

$$\left[ \frac{\partial \Omega}{\partial \mu} \right] = -Ne$$

Non-relativistic particles

$$\epsilon = \frac{p^2}{2m}; \quad m = mc \quad : \quad PeV = \frac{2}{3} E$$

Relativistic particles:

$$\epsilon = cP \quad : \quad PeV = \frac{1}{3} E$$
$\frac{1}{e^{(E-\mu)/kT} + 1}$

$T \text{ "small"}$

$T \text{ "large"}$

$\mu = \varepsilon$

$\frac{Nv}{V} \sim \int_0^{P_0} p^2 dp \sim P_0^3$; $P_0: \varepsilon = \mu$

$\frac{E}{V} \sim \int_0^{P_0} \varepsilon p^2 dp$

$\int_0^{P_0} p^4 dp \sim P_0^5$ non-relativistic

$\int_0^{P_0} p^3 dp \sim P_0^4$ relativistic

$P_e \sim \frac{E}{V}$

$P_0 \sim \left(\frac{Nv}{V}\right)^{5/3} \sim \varepsilon^{5/3}$ non-relativistic

$P_0 \sim \left(\frac{Nv}{V}\right)^{4/3} \sim \varepsilon^{4/3}$ relativistic

$T \text{ "small": degeneracy}$

Complete degeneracy:
"T small", condition for degeneracy

\[ \hbar T \ll \varepsilon_0 \]

\[ \varepsilon_0 \approx p_0 \text{ maximum momentum or energy} \]

Non-relativistic particles:

\[ \varepsilon_0 \approx p_0^2 \sim \left( \frac{N e}{V} \right)^{2/3} \]

Condition for degeneracy:

\[ \hbar T \ll \frac{k^2}{m} \left( \frac{N e}{V} \right)^{4/3} \]

Valid for all Fermions: nuclei degenerate at lower temperatures than electrons \( \rightarrow e^- \)-dominate

Degeneracy:

EOS independent of \( T \)

\[ p_e \propto \rho^{5/3} \quad \text{non-relativistic } e^- \]

\[ p_e \propto \rho^{4/3} \quad \text{relativistic } e^- \]
Strahlungsdruck
\[ p = p_{\text{val}} \]
\[ \beta \rightarrow \sigma \]

Ideales Gas
\[ \alpha = 5\beta + 1 \]

Nicht relativ - relativ

Entartung
\[ \alpha = \frac{3}{2}, \quad s = 0 \]

\[ (D) + (A) \]
\[ \alpha = \frac{3}{3}, \quad s = 0 \]
Opacity

Sources

- Electron scattering (Thomson scattering)
  
  cross section $\sigma_e = \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2}\right)^2$
  
  constant, frequency independent
  
  $\kappa_e = \sigma_e \frac{m_e}{5} = 0.2 \frac{\text{cm}^2}{g} (1 + x)$
  
  (fully ionized matter)

- Free–free transitions

- Photo-ionization
  
  Bound–free transitions

- Bound–bound transitions

- H⁻ opacity for $T < 6000 \text{K}$
  
  $H + e^- \leftrightarrow H^-$

- Conduction if electrons are degenerate
Treatment of Opacity

For analytical estimates: Kramer's opacity

Power law fit:

$$K(g, T) \sim g^\alpha T^\beta$$

with, e.g., $\alpha = 1$ and $\beta = -3.5$

Opacity strongly dependent on chemical composition

For realistic calculations: Opacity tables

Table parameters: $g$, $T$, composition

Projects: LAOL: Los Alamos
          OPAL: Livermore
          OP: MHD
electron scattering

log T (K)

\[ x_a, 10x_a, 10^{2}x_a, x_{bf} = x_f, 10^x_k, 10^{3}x_a, 10^{4}x_a \]

\[ \log \rho (g/cm^2) \]

conduction by degenerate electrons

\[ a = 0 \]

Kappa (cm^2 g^{-1})

Log T (K)

\[ 0.0, 1.0, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0 \]

\[ 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5, 8.0 \]
Consider reaction:

\[ A + B \rightarrow C + D \]

Energy gained due to mass difference:

\[ m_A + m_B \neq m_C + m_D \]

Energy release:

\[ \Delta Q = \Delta m c^2 \]

Binding energy:

\[ BE_i = \left[ 2m_\text{p} + (A-2)m_\text{n} - m_i(A,Z) \right] c^2 \]

energy released by forming nucleus \((A,Z)\)

Binding energy per nucleon:

\[ BE/A := \ell \]

\(\Delta Q\) corresponds to difference in binding energies:

\[ \Delta Q = BE_C + BE_D - BE_A - BE_B \]

\[ = A_c \ell(A_c) + A_D \ell(A_D) - A_A \ell(A_A) - A_B \ell(A_B) \]

\(\Delta Q > 0\) only for \(A_A + A_B \leq 56\) (Fe)

\(\Delta Q < 0\) otherwise
Fig. 10  Bindungsenergie pro Nukleon als Funktion von A für stabile Kerne; nach [Eva 55]. Abszisse bis A = 30 gespreizt.
Nuclear Potential

\[ V \]

\[ E_c \]

\[ E_c \sim \text{MeV} \]

\[ E_{\text{kin}} \sim \frac{3}{2} kT \sim \text{keV} \]

\[ kT \sim 10^3 K \]

\[ \text{strong interaction} \]

\[ \text{attractive forces} \]

\[ \text{Overcoming of Coulomb barrier} \]

\[ dN \sim e^{-E/kT} \]

\[ \text{Maxwell distribution} \]

\[ \text{particles also at high energies} \]

\[ \text{tunneling: finite potential barrier} \]

\[ \text{probability even for } E < E_c \]

Nuclear Astrophysics: low energy domain

\[ E_{\text{kin}} \ll E_c \]
Reactions rate $r_{12}$ (reactions/Volume/fixed)

of particle 1 (at rest) with particle 2

$n_{12}$: number density

$v$: (relative) velocity

$j_2$: flux of particle \( j_2 = n_2 v \)

\[ r_{12} = n_1 j_2 \sigma_{12} = n_1 n_2 v \sigma_{12}(v) \]

\( \sigma_{12} \): cross section

velocity distribution for particles \( W(v) \)

\[ \int W(v) \, dv = 1 \]

\[ r_{12} = n_1 n_2 \int W(v) v \sigma_{12}(v) \, dv = n_1 n_2 \langle v \sigma_{12} \rangle \]

\[ \langle v \sigma_{12} \rangle \Rightarrow \chi_1, \chi_2, \varepsilon_{12} \]

\[ \varepsilon_{12} = \frac{\Delta Q_{12} \cdot r_{12}}{\sigma} = \sigma \frac{\chi_1 \chi_2}{m_1 m_2} \Delta Q_{12} \langle v \sigma_{12} \rangle \]

identical particles: sect \( \frac{1}{2} \)

\( \sigma_{12} \): if possible, experimentally

problem: low energy, extrapolation necessary
Theoretical treatment of $\sigma$

$\sigma \sim P_R P_c$

Recoil probability $P_R \sim 1/E$

Tunneling probability $P_c \sim \exp\left(-\frac{2a z_1 z_2 e^2}{\hbar v}\right)$

$\sigma(E) = \frac{S(E)}{E} \exp\left(-\frac{2a z_1 z_2 e^2}{\hbar v}\right)$

$S(E)$ constant: non-resonant reaction

$E = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v^2$

Velocity distribution: Maxwell

$w(v) dv = w(E) dE \sim \frac{E^{3/2}}{(\kappa T)^{3/2}} \exp\left(-\frac{E}{\kappa T}\right) dE$

$\langle \sigma v \rangle = \int w(v) \sigma(v) v dv$

$\sim \frac{A}{(\kappa T)^{3/2}} \int dE S(E) \exp\left(-\frac{E}{\kappa T} - \frac{E}{\sqrt{2E}}\right)$

![Diagram](image)
Approximation of Gamow-peak:

\[ \exp \left( -\frac{E}{kT} - \frac{\Delta}{\sqrt{E}} \right) \approx e^{-\Delta} \exp \left[ -\left( \frac{E-E_0}{\Delta/2} \right)^2 \right] \]

\[ \Delta = \frac{4}{\sqrt{3}} \left( E_0 kT \right)^{1/2} \quad ; \quad \tau = \frac{3E_0}{kT} \sim T^{-1/3} \]

\[ \langle \sigma v \rangle \sim \frac{\sigma(E_0)}{k} \tau^2 e^{-\Delta} \]

\[ \tau_{12} \sim \tau e^{-\Delta} \]

Sometimes parametrization of \( \tau \) as

\[ \tau \sim \rho T \nu \quad \nu \sim 5...40 \]
Hydrogen Burning

\[ ^4p \rightarrow ^4\text{He} + 2e^+ + 2\nu \]

\( T \leq 1.8 \times 10^7 \text{K} \) : pp-chain (5)

\[
\begin{align*}
^4\text{H} + ^4\text{H} &\rightarrow ^2\text{D} + e^+ + \nu \\
^2\text{D} + ^4\text{H} &\rightarrow ^3\text{He} + \gamma \\
^3\text{He} + ^3\text{He} &\rightarrow ^4\text{He} + 2^1\text{H} \\
^3\text{He} + ^4\text{He} &\rightarrow ^7\text{Be} + \gamma \\
^7\text{Be} + e^- &\rightarrow ^7\text{Li} + \nu \\
^7\text{Li} + ^4\text{H} &\rightarrow 2^4\text{He} \\
^7\text{Be} + ^4\text{H} &\rightarrow ^8\text{B} + \gamma \\
^8\text{B} &\rightarrow ^8\text{Be}^* + e^+ + \nu \\
^8\text{Be}^* &\rightarrow 2^4\text{He}
\end{align*}
\]

\( \beta \) decay determines, fine scale and energy generation

Element reaction: \(^4\text{H} (p, e^+\nu) ^2\text{D} \)
$T \gtrsim 1.8 \times 10^7 K$ : CNO-cycle(s)

3 cycles, two of them:

Primary cycle:

- $^{12}C + ^{4}H \rightarrow ^{13}N + \gamma$
- $^{13}N \rightarrow ^{13}C + e^+ + \nu$
- $^{13}C + ^{4}H \rightarrow ^{14}N + \gamma$
- $^{14}N + ^{4}H \rightarrow ^{15}O + \gamma$
- $^{15}O \rightarrow ^{15}N + e^+ + \nu$
- $^{15}N + ^{4}H \rightarrow ^{12}C + ^{4}He$

Secondary cycle:

- $^{15}N + ^{4}H \rightarrow ^{16}O + \gamma$
- $^{16}O + ^{4}H \rightarrow ^{17}F + \gamma$
- $^{17}F \rightarrow ^{17}O + e^+ + \nu$
- $^{17}O + ^{4}H \rightarrow ^{16}N + ^{4}He$

CNO : catalyst

Element reaches $^14N(p, \gamma)^{15}O$

control, timescale and energy generation
Helium burning \[ T \approx 1.2 \times 10^8 \text{K} \] (Novak & Sneden)

3α-reaction (Salpeter)

\[ 3^4\text{He} \rightarrow ^{12}\text{C} \]

Occurs in 2 steps:

\[ ^4\text{He} + ^4\text{He} \rightarrow ^8\text{Be} \quad T_{^8\text{Be}} \approx 3 \times 10^{-16} \text{sec} \]

\[ ^8\text{Be} + ^4\text{He} \rightarrow ^{12}\text{C}^* \]

By-reaction produce \[^{16}\text{O}\] for

Carbon burning ashes: \[^{20}\text{Ne}, \ ^{23}\text{Na}, \ ^{24}\text{Mg}\]

Several reactions:

\[ ^{12}\text{C} (^{12}\text{C}, \alpha) ^{20}\text{Ne} \]

\[ ^{23}\text{Na} (p, \gamma) ^{24}\text{Mg} \]

\[ T \approx 5 \ldots 7 \times 10^8 \text{K} \]

Neon burning ashes: \[^{16}\text{O}, \ ^{24}\text{Mg}, \ ^{28}\text{Si}\]

Most important reactions:

\[ ^{20}\text{Ne} (\alpha, \gamma) ^{16}\text{O} \]

\[ ^{24}\text{Mg} (\alpha, \gamma) ^{28}\text{Si} \]

\[ T \approx 10^9 \text{K} \]
Oxygen burning ashes: mainly $^{28}\text{Si}$
very complicated reaction network
example:

$$^{16}\text{O} (^{16}\text{O}, \alpha)^{28}\text{Si}$$
$$^{34}\text{P}$$

$$^{31}\text{P} (\text{e}^+\nu)$$

$T \approx 1.2 \ldots 1.5 \times 10^9 \text{ K}$

Silicon burning ashes: iron group nuclei
e.g. $^{56}\text{Fe}$
very complicated reaction network
transition to "nuclear statistical equilibrium" (NSE)

"netto" reaction:

$$^{28}\text{Si} + ^{28}\text{Si} \rightarrow ^{56}\text{Fe}$$

$T \approx 2 \times 10^9 \text{ K}$
Emit stellar models with prescription
\[ p = \kappa s^{1+\nu} \]
\( \nu: \) polytropic index

Justifications:
- equation of state \( p = p(s) \)
  \( \text{e.g.: degenerate Fermiions} \)
- efficient energy transport \( \Rightarrow T = \text{constant} \)
  \[ p = p(s) \]
- efficient convection: \( \nabla = \nabla_{ad} \)
  \[ p = p(s, s) = p(s) \]

Mechanical equations
\[ \frac{1}{s} \nabla p = -\nabla \Phi \quad \text{hydrostatic equilibrium} \]
\[ \Delta \Phi = 4\pi G p \quad \text{Poisson} \]

System closed by polytropic relation,
energy equations disregarded
Spherical symmetry
\[ \frac{1}{s} \frac{dp}{dv} = -\frac{d\Phi}{dv} \quad (1) \]
\[ \frac{1}{r^2} \frac{d}{dv} (r^2 \frac{d\Phi}{dv}) = 4\pi G p \quad (2) \]
Insert (1) into (2), replace $p$ by polytropic relation

$$-K (n+1/m) \frac{\alpha}{v^2} \frac{d}{dr} (r^2 \rho^{m-1} \frac{d\rho}{dr}) = 4\pi G \rho$$

Normalization and transformation

$$s = s_c \Theta^n$$
$$r = \alpha s$$
choose $\alpha = \frac{K (m+1)}{4\pi G} s_c$

$$\frac{1}{s^2} \frac{d}{ds} (s^2 \frac{d\Theta}{ds}) = -\Theta^n$$
Lame–Emden equation

$$\Theta (s=0) = 1 ; \frac{d\Theta}{ds} (s=0) = 0$$ (regularity)

single parameter: $n$

$n = 0, 1, 5$: analytical solutions

$n < 5$: $\Theta = 0$ for $s < s_0$
with zero $\int_0^{s_0} (n)$ shell boundary

Mass:

$$M = \int_0^R s \rho \pi r^2 dr = \alpha^3 4\pi s_c \int_0^{s_0} \Theta^n s^2 ds$$

$$M_n (n) = -s_c \frac{d\Theta}{ds} |_{s_0}$$

Radius:

$$R = \alpha \int_0^{s_0}$$

Mean density:

$$\langle s \rangle = \frac{M}{4\pi \frac{R^3}{3}} = -3 s_c \frac{d\Theta}{ds} |_{s_0}$$
mean density, definition of $\sigma$, mass $\Rightarrow$

**Mass - Radius - Relation for Polytopes:**

\[
R^{3-n} = \frac{4}{3\pi} \left( \frac{K(n+2)}{G} \right)^n \left[ -f_{\omega}^{n+1\omega} \left[ \frac{d\Theta}{df} \right] \right]^{n-1} M^{1-n}
\]

\[R^{3-n} \sim M^{1-n}\]

$n = 3$  \hspace{1cm} $M$ independent of $R$

$n = 1$  \hspace{1cm} $R \sim M$

\[
\frac{dR}{dM} \sim \frac{1-n}{3-n} M - \frac{2}{3-n}
\]

\[
\frac{dR}{dM} > 0 \hspace{1cm} n < 1, \ n > 3
\]

\[
\frac{dR}{dM} < 0 \hspace{1cm} 1 < n < 3
\]

$n = 3/2$ : \hspace{1cm} $R \sim M^{-4/3}$

$n = 3$ : \hspace{1cm} Mass fixed by value of $K$
Lane–Emden–Funktionen

\[
\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy}{dx} \right) = -y^n
\]

\[
x = \xi, \quad y = \Theta
\]
Stars form from interstellar medium (ISM)

\[ \text{ISM} \lesssim \text{Dust} \lesssim \text{Gas} \lesssim \text{H}_2 \text{; } T \approx 100 \text{ K} \lesssim \text{H} \text{; } T \approx 10^4 \text{ K} \]

Typical density \( \approx 100 \text{ atoms/cm}^3 \), 0.1-1

Gravitational instability of interstellar cloud:

\[ \left| \frac{4\pi}{3} \frac{dP}{dr} \right| < \left| \frac{GM^2}{r^2} \right| \]

\[ P]/\rho \approx \frac{R}{\mu} T < \frac{GM}{R} \]

\[ M > \left( \frac{RT}{\mu G} \right)^{3/2} \frac{1}{\sqrt{5}} \equiv M_{\text{y}} \text{, } \text{M}_{\odot} \text{, } \text{yr} \]

\text{Year's criterion}

\( T = 100 \text{ K} \; \Rightarrow \; \rho = m_p \cdot 400/\text{cm}^3 \Rightarrow M_y \approx 3000 \text{ M}_\odot \)

Isothermal collapse: \( \rho \uparrow \Rightarrow M_y \downarrow \)

\( \Rightarrow \text{smaller masses become unstable} \Rightarrow \text{fragmentation} \)

\text{Consequence:}

Stars form in clusters.
3 Phases:

1) Isothermal collapse, optically thin

2) Star becomes optically thick, \( T \uparrow \)
   - Pressure stops collapse
   - Hydrostatic core forms

3) Envelope continues to fall onto hydrostatic core: "accretion phase"
   - \( M < 3 M_\odot \): accretion phase finished before nuclear reaction starts

   \( M > 3 M_\odot \): accretion continues; when nuclear reaction starts

Conceit: These phases of star formation still poorly understood

Characteristics at the end of phase 3:

- Low temperature (\( \leq 2000 \text{ K} \)) \( \rightarrow \) high opacity
- High densities
- Objects are fully convective
- Chemical homogeneity

Stellar evolution starts chemically homogeneous
High density $\rightarrow$ Convection efficient

Stratification adiabatic: $\nabla = 0$

Properties of matter:

\[
p = \frac{R}{\mu} s T \\
c_v = \frac{3}{2} \frac{R}{\mu}
\]

\[
TdS = 0 = c_v dT - \frac{p}{\rho^2} ds
\]

\[
0 = \frac{3}{2} \left( \frac{4}{3} dp - \frac{p}{\rho^2} ds \right) - \frac{p}{\rho^2} ds
\]

\[
\frac{dp}{p} = \frac{5}{3} \frac{ds}{s} \\
p = K s^{5/3} \\
p = K^1 T^{5/2}
\]

Shells thinner described by polytrope $n = 3/2$

\[
K = R^{\frac{3-n}{n}} M^{-\frac{4-n}{n}}
\]

\[
K^1 \sim M^{-1/2} R^{-3/2}
\]

\[
p = K^1 T^{5/2}
\]

Photosphere:

\[
P_{\text{eff}} = \frac{2}{3} \frac{GM}{R^2} \frac{1}{\sqrt{L_{\text{eff}}}}
\]

\[
L = 4\pi R^2 \sigma T_{\text{eff}}^4
\]

Require $p = P_{\text{eff}}$, $T = T_{\text{eff}}$

Eliminate $R$, $p = P_{\text{eff}} \Rightarrow T_{\text{eff}}(L, M)$

Approximation:

\[
\log T_{\text{eff}} \approx \frac{L}{80} \log L + \frac{1}{6} \log M + \text{constant}
\]

Hayashi - Livi ni HRD
Meaning of the Hayashi-line

HL separates domains with $\bar{\epsilon} < \bar{\epsilon}_{\text{ad}}$ and $\bar{\epsilon} > \bar{\epsilon}_{\text{ad}}$ in the HRD by curving in:

$\bar{\epsilon} < \bar{\epsilon}_{\text{ad}}$: star partially radiative (left to HL)

$\bar{\epsilon} = \bar{\epsilon}_{\text{ad}}$: HL

$\bar{\epsilon} > \bar{\epsilon}_{\text{ad}}$: right to HL

- Convection extremely strong
- Reduces $\bar{\epsilon}$ to $\bar{\epsilon}_{\text{ad}}$
- Reach in dynamical timescales
- No hydrostatic equilibrium

"forbidden region"

Star formation ends on HL.

First hydrostatic configuration on HL.

Fully convective stars left and evolve on HL.

HL in the right boundary of HRD.

Region right to HL forbidden for hydrostatic stars.
Evolution after reaching Hb:

Source of energy: \( E_6 \)

Time scale of evolution: \( T_{KH} \)

Virial theorem: \( T \uparrow \)

Pre-main sequence phase finished if nuclear burning starts

= definition of "zero age main sequence" ZAMS

ZAMS: Chemical composition homogeneous

"Primordial" \( X \approx 0.7, Y \approx 0.28, Z \approx 0.05 \)

[\( X \approx 0.75, Y \approx 0.25, Z = 0 \)]

Main sequence: Location of H-burning

Phase of H-burning
Main sequence

MS: H-burning by definition

Timescale close
Thermal equilibrium \( \frac{dP}{d\nu} = 0, \frac{dT}{d\nu} = 0 \)
Parameters: \( \mu, M \)
Position of MS: \( L(T_{eff}) \) ?

Estimates

EOS: \( P = \frac{R}{\mu^2} gT \)
Energy generation \( E \sim g \nu^5 \)

\[
\begin{align*}
\frac{x}{M_\nu} & \sim \frac{\Lambda}{\nu^2 g} \\
\frac{P}{M_\nu} & \sim \frac{M_\nu}{\nu^4} \\
\frac{P}{g} & \sim \frac{T}{\mu} \\
T/M_\nu & \sim \frac{L_r \nu}{\nu^4 T^3} \\
L_r & \sim \frac{\Lambda}{\nu^4 M_\nu^3}
\end{align*}
\]

Mass - luminosity - relation

Energy equation?
Massive stars evolve faster
Energy equation:
\[ L \propto M v E \propto M v^3 T^v \propto M v^{\frac{v-4}{v+3}} \left( \frac{M v}{v} \right)^{\frac{v-1}{v+3}} \]

Mass - Radius - relation:
\[ r \propto \mu^{\frac{v-1}{v+3}} M_r \]

Main sequence:
\[ v \approx 1.3 \]
\[ R \propto M^{3/4} \]
\[ L \propto M^3 ; \quad R \propto L^{1/4} \]
\[ L \propto R^2 T_{\text{eff}}^4 ; \quad L^{1/2} \propto T_{\text{eff}}^4 \]
\[ \log L \approx 8 \log T_{\text{eff}} + \text{constant} \quad \text{function of } v \]

Mass varies along main sequence.
Comparison: \[ R = \text{constant} ; \]
\[ \log L \approx 4 \log T_{\text{eff}} + \text{constant} \]
Evolution on MS during H-burning

Assume fixed mass, $\mu$ changes

\[ L \sim \mu^4 \]

\[ R \sim \mu^{\frac{\nu-4}{\nu+3}} \]

Stefan-Boltzmann:

\[ \log L \sim \frac{\nu+3}{\nu+10} \quad 8 \log T_{\text{eff}} + \text{constant} \]

Maß sequence

\[ \log L \sim \quad 8 \log T_{\text{eff}} + \text{constant} \]

Evolution would proceed to the left of MS!

Numerical results: stars evolve off to the right!

Reason: Chemical inhomogeneity

Timescale

\[ \frac{T_{\text{H} \rightarrow \text{He}}}{T} \sim 0.8 \ldots 0.9 \]

H-burning longest timescale, most stars on MS

\[ T_{\text{H} \rightarrow \text{He}} \sim \frac{M}{L} \sim \frac{1}{M_\odot} \times 6 \times 10^9 \left( \frac{M}{M_\odot} \right)^{-2} \text{ year} \]

Stars evolve off the MS to the right.

Higher masses reach (hardly) them low masses

\[ \Rightarrow \text{Occurrence of a "knee" in the size-mass of clusters} \]
Internal structure on MS

2. Sections

\[ M \geq 1.5 M_\odot \]
- CNO-cycle(s)
- \( V \) large
- core convection
- envelope convection

\[ M \leq 1.5 M_\odot \]
- pp-chain(s)
- \( V \) small
- core radiation
- envelope convection

onset of convection
controlled by opacity and energy generation

End of Main sequence evolution: hydrogen exhaustion in the stellar core
Post-Main Sequence Evolution

Scheme

nuclear fuel exhausted
\[\downarrow\]
ignition of next burning

\[\rightarrow\]
contraction
\[\uparrow\]

\[\rightarrow\]
temperature increase
\[\rightarrow\]
virial theorem

Structure of burning:
Core
Shell

Result:
Generation of onion-skin structure

Within a burning phase is reached
Sensitivity depends on mass
Fig. 5A. Schematic illustration (not to scale) of the 'onion-skin' structure in the interior of a newly-evolved massive star (25$M_\odot$). Numbers along the vertical axis show some typical values of mass fraction, while those along the horizontal axis indicate temperatures and densities (g m$^{-2}$). Adapted from R. Kippenhahn & A. Weigert, Stellar Structure and Evolution. Springer-Verlag 1990.
Consider homologous contraction with time

Homologous: relative structure conserved

\[
\begin{align*}
\frac{\dot{\rho}}{\dot{\rho}} &= -3 \frac{x}{r} \\
\frac{\dot{P}}{P} &= -4 \frac{x}{r} \\
\frac{\dot{P}}{P} &= \alpha \frac{\dot{P}}{P} - \delta \frac{\dot{T}}{T}
\end{align*}
\]

Hydrostatic equilibrium

\[
\frac{\dot{P}}{P} = \frac{4\alpha - 3}{3\delta} \frac{\dot{\rho}}{\dot{\rho}}
\]

Equation of state

Ideal gas: \( \alpha = \delta = 1 \)

\( T \sim \rho^{1/3} \)

Constant degree of degeneracy:

\( T \sim \rho^{4/3} \)

\( \Rightarrow \) Degree of degeneracy increases during contraction

Complete non-relativistic degeneracy:

Temperature independence: \( \delta = 0 \)

Pressure - density relation: \( \alpha = 3/5 \)
During contraction $\delta$ vanishes from 1 to 0
$\alpha$ vanishes from 1 to 3/5

$$\frac{\dot{T}}{T} > 0 \quad \text{for } \alpha > 3/4$$
$$\frac{\dot{T}}{T} < 0 \quad \text{for } \alpha < 3/4$$

Degeneracy: $T$ decreases for contraction.

Consequence: lower burning phases will not be reached if degeneracy sets in.

$T$ maximum temperature $T$
During transition and contraction from ideal non-degenerate matter to degenerate electron gas.

At which stage degeneracy is reached, depends on mass.

Variational theorem?
Degeneracy before H-burning: \( M < 0.08 \, M_\odot \)

- \( \text{He} \) : \( M_{\text{He}} < 0.35 \, M_\odot \)
- \( \text{C} \) : \( M_{\text{C}} < 0.9 \, M_\odot \)

Violation of hemology: shell burning

Temperature increase in degenerate region

\( \Rightarrow \) Ignition of burning under degenerate conditions

\( \text{He} - \text{flash} \quad \text{C} - \text{flash} \)

Burning under degenerate conditions unstable

Degeneracy: \( p = p(\rho) \)

Nuclear energy release

\( dQ = c_v \, dT \)

Extrême rise of temperature

\( \varepsilon \sim T^\nu \)

Overproduction of nuclear energy

\( \Rightarrow \) Instability, "flash"

Final product of shell evolution

Degenerate He cores: \( M_{\text{initial}}/M_\odot \lesssim 2.5 \)

- \( \text{C/O} \) : \( 2.5 \lesssim M_{\text{final}}/M_\odot \lesssim 8 \)
Evolutionary tracks
Interior Structure

Representative models: $M = 5M_\odot$ and $M = 7M_\odot$

Characteristics

Evolution from main sequence to red giant
Turnscales, hydrogen burning - gap
Several "loops", crossing of Cepheid strips

Internal structure, core - shell burning, convective zones

Minor principle at shell burning, core contraction - envelope expansion

Why do stars become red giants?
$M = 5M_\odot$

- Major phase of core helium burning ($9 \times 10^4$ yr)
- Establishment of shell source ($1.4 \times 10^4$ yr)
- Hydrogen burning in thick shell ($1.2 \times 10^4$ yr)
- Overall contraction phase ($2.2 \times 10^4$ yr)
- Hydrogen burning in core ($6.44 \times 10^7$ yr)
- Shell-narrowing phase ($8 \times 10^3$ yr)
- Convective envelope begins to extend inward rapidly—surface abundances begin to change
- Red-giant phase ($5 \times 10^4$ yr)
- Core neutrino loss, thin helium-burning shell
- Thick helium-burning shell
- Overall contraction with exhaustion of central helium
- Ignition of triple-$\alpha$ process
- First phase of core helium burning ($6 \times 10^4$ yr)
- Disappearance of deep convective envelope, rapid contraction ($10^4$ yr)
\[ M = 7 M_\odot \]
Late stages of evolution

$M \leq 8 \ldots 10 \, M_\odot$

Degenerate cores
Red Giants

Mass loss, Pulsation
Main - variable

"Supermain"

Shell does envelope:
Planetary nebula

\[\text{log } L \uparrow \quad \text{MS} \quad \text{HL}\]
\[\text{log } T_{\text{eff}} \downarrow\]

Degenerate core appears as He-, C/O - white dwarf

$M \geq 8 \ldots 10 \, M_\odot$

All burning phases are reached, Si - burning
forms Fe - Ni - core
Consider polytrope with \( n = 0 \): 

\[
\frac{\frac{d P}{d r}}{\frac{d r}{r}} = - \frac{G M r}{r^2}
\]

\( n \) constant: 

\[
M r = \frac{4 \pi}{3} r^3 P
\]

\[
\frac{\frac{d P}{d r}}{\frac{d r}{r}} = - \frac{4 \pi}{3} G \frac{P}{r^2}
\]

Boundary condition: \( P = 0 \) for \( r = R \)

\[
P = \frac{2 \pi}{3} G \frac{p}{r^2} \left( R^2 - r^2 \right) = \frac{2 \pi}{3} G \frac{p}{r^2} \left( 1 - \left( r/R \right)^2 \right)
\]

Consider homologous variation of radius 

(radius structure is not varied; \( r/R \) constant)

\[p \propto R^{-4} \] (each point)

Pressure - Radius relation for equilibrium

Pressure provided by thermal pressure

Assumption: Homologous variation sufficiently fast \( \Rightarrow \) changes of state adiabatic

\[
p v s \text{ Yad} \sim R^{-3} \text{ Yad}
\]

Equilibrium only for \( 3 \text{ Yad} = 4 \)
3 $\gamma_{ad} > 4$ Thermal pressure increases faster than equilibrium value => reducing force, stability

3 $\gamma_{ad} < 4$ Thermal pressure increases more slowly than equilibrium value, pressure deficit, contraction will be amplified, instability

Criteria for stability of mechanical equilibrium of a star (dynamical stability)

$\gamma_{ad} > 4/3$

$\gamma_{ad} < 4/3$ star unstable on dynamical timescales, enter dynamical phase

Exact analysis ($\gamma_{ad}(r)$) $\gamma_{ad} = \frac{\int_0^R r^2 \gamma_{ad} \rho \, dr}{\int_0^R r^2 \rho \, dr} < 4/3$

Sufficient for instability
White dwarfs

Final stage of stellar evolution

Bremsstrahlung provided by degenerate electrons

\[ p \sim \mu_e^{\frac{5}{3}} \frac{5}{3} \]

non-relativistic electrons

\[ p \sim \mu_e^{-\frac{4}{3}} \frac{4}{3} \]

relativistic electrons

Polytrope's

\[ R \sim M^{\frac{1-n}{3-n}} \]

\[ \rho_c \sim M^{\frac{2n}{3-n}} \]

non-relativistic electrons: \( n = \frac{3}{2} \)

relativistic electrons: \( n = 3 \)

Non-relativistic electrons:

\[ R \sim M^{-\frac{1}{3}} \]

\[ \rho_c \sim M^{2} \]

Mass-Radius relationship of white dwarfs

Increasing \( M \rightarrow \rho_c \uparrow, \frac{N_e}{V} \uparrow, \frac{N_e}{V} \sim p_0^3 \)

\[ p_0 \uparrow, E_{\text{max}} \uparrow \]

Consequence: With increasing \( M \), fermionic \( \leftrightarrow \) relativistic electrons
Consequence: Mass fixed

\[ M = M_{\text{ch}} = \frac{5.76 \, M_{\odot}}{\mu_e^2} \]

Apart from \( \mu_e \) the Chandrasekhar limit depends only on fundamental constants, \( \mu_e = 2 \) for \(^{4}\text{He}, ^{12}\text{C}, ^{16}\text{O}, ^{20}\text{Ne}, ^{24}\text{Mg} \):

\[ M_{\text{ch}} = 1.44 \, M_{\odot} \]

Physical meaning:

- \( M = M_{\text{ch}} \): No hydrostatic equilibrium
- \( M < M_{\text{ch}} \): Pressure dominates gravity, star expands \( \rightarrow \) non-relativistic deg. \( \rightarrow \) equilibrium
- \( M > M_{\text{ch}} \): Gravitational dominance, pressure star collapses

\( M_{\text{ch}} \) is the maximum mass for (electronic) degenerate stars, i.e., for white dwarfs.
Scenarios

1) - Binary systems containing WD (He, C/O, ...)
   - Mass transfer to WD
   - Collapse when $M = M_{\text{CH}}$ is reached
   - Ignition of He-, C/O-burning under degenerate conditions, instability
   - Thermal runaway, thermonuclear explosion
   - Probably disruption details?
   - Production of heavy elements ($^{56}$Fe, ...)
   - Supernova I, standard candles (?)

2) - Massive star reaching Si-burning
   - Due to lack of exo-burning nuclear reactions
     Fe-Ni core contracts and becomes degenerate
   - Si-burning increases Fe-Ni core until $M = M_{\text{CH}}$ is reached or
     $M_{\text{CH}}$ is reduced by inverse $\beta$ decay
     $$ p + e^- \rightarrow n + \nu $$
   - Collapse, no thermonuclear explosion
   - Photodissociation of nuclei; e.g.,
     $$ ^{56}_{26}\text{Fe} \rightarrow 30\,n + 26\,p $$
     $$ \text{inverse } \beta \text{ decay} $$
     $$ p + e^- \rightarrow n + \nu $$
- Photodissociation minima
  $\delta_{\nu M} < 4/3$
- Dynamical instability
- Collapse stopped by "neutron primer"
- Formula of neutron star
- Timescale for collapse in sec
- Envelope collapses onto neutron star
- Shock wave, explosion, SN II
- Energy balance
  - Gravitational energy released $\sim 10^{53}$ erg
  - Gravitational energy of envelope $\sim 10^{50}$ erg
  - Energy of shock wave $\sim 10^{51}$ erg
  - Radiations $\sim 10^{54}$ erg
  - Neutrinos $\sim 10^{52-53}$ erg
Neutron Stars

Formation by Supernovae
e.g. Crab Nebula

Pressure (first approximation): Degeneracy of neutrons

TOV limit: analogous to Chandrasekhar limit

General relativistic correction ≈ 20%
Radii: ~ 10 km

Problem: Equation of state: Nuclear densities

Magnetic fields: \( B \lesssim 10^{12} G \)
Rotation: \( P \approx 1 \text{ msec} \)

\( M - \mathcal{E} \) diagram:

\[
\frac{dM}{d\mathcal{E}} > 0 \iff \mathcal{E}_{\text{adm}} > \frac{4}{3} \quad \text{stability}
\]

\[
\frac{dM}{d\mathcal{E}} < 0 \iff \mathcal{E}_{\text{adm}} < \frac{4}{3} \quad \text{instability}
\]
Oscillations and Pulsations of Stars

The star as an acoustic cavity with eigenfrequencies, modes etc analogous to organ pipe

\[ \nu \cdot A = c_{\text{sound}} \]

organ pipe: \[ \lambda = \frac{L}{n} \]

star: \[ \lambda = \frac{R}{m} \]

\( m: \) number of nodes, order of overtone

\[ c_{\text{sound}} = \frac{y_{\text{ac}} P}{\rho} \]

Eigenfrequencies of a star

\[ \nu \sim \frac{M}{R} \sqrt{\frac{y_{\text{ac}} P}{\rho}} \]

Hydrostatic equilibrium

\[ \frac{d}{dx} \left( \frac{1}{2} \rho \nu^2 \right) = -\frac{GM:\nu}{r^2} \]

Estimate

\[ P/\nu \sim \frac{GM}{R} \]

\[ \nu \sim m \left( \frac{y_{\text{ac}} G \rho}{\nu} \right)^{1/2} \]
Fundamental mode: \( n = 1 \)

Period \( \Pi \):

\[
\Pi \sqrt{\frac{G}{\mu}} = \text{constant}
\]

Period - density - relation

\[
\Pi \left( \frac{\rho}{\bar{\rho}_0} \right)^{1/2} = \Phi \ ; \quad 0.03 \leq \Phi \leq 0.12
\]

Excitation of pulsation by Cantor-type process

K-mechanism: require special temperature dependence of \( \kappa \)

compression phase: \( \kappa \) larger

clumping up radiation \( \rightarrow \)

excess pressure \( \rightarrow \) excitation

K-mechanism restricted to narrow temperature range: \( \text{Teff} (L) \) Cepheid - strip

Stefan - Bolzmann law: Mass - luminosity - relation

\[
L \sim R^2 \text{Teff}^4
\]

\( \Rightarrow R(L), M(L) \)

with \( \Pi \sqrt{\frac{G}{\mu}} = \text{constant} \)

Period - luminosity - relation

\[
\Pi(L)
\]

Cepheids: \( \Pi \sim 0(d) \) : important distance indicators
- $^4$He: primordial
- iron group elements and below: thermonuclear fusion in stars
- Synthesis of elements heavier than iron:
  - Neutron capture (building energy per nucleon, contours burn)
- Abundances: double maxima:
  - $r$ process: rapid neutron capture
  - $s$ process: slow, giants
- Competition between $\beta$ decay $\leftrightarrow n$ capture
- Number of nuclei exposed to neutron source
  \[ N \sim N_0 \times \sigma \] where $\sigma$ is cross section for capture
- $s$-process: capture slow compared to decay
  - proceeds via "stability valley"
- $r$-process: proceeds outside "stability valley"
  - step of exposure $\rightarrow \beta$ decay
  - distribution shifted towards smaller number of neutrons

Fig. 6.9. Neutron capture paths in the N,Z plane. The r-process path was calculated for a temperature of $10^8$ K and a neutron density of $10^{14}$ cm$^{-3}$ (Seeger, Fowler & Clayton 1965). The dotted curve shows a possible location of the neutron drip line after Uno, Tachibana & Yamada (1992). Adapted from Rolfs & Rodney (1988).
Point sources of radiation (Stars)

\[ \frac{L}{4\pi D^2} = F \]

\( L \): luminosity \quad \text{\( D \): distance} \)

Apparent brightness (magnitude):

\[ m \sim \log F \]
\[ m = -2.5 \log F + \text{constant} \]

Dependence on wavelength:

\[ m_U, m_B, m_V, m_R, m_I \]

Fiskis: \( U, B, V, R, I \)

For total energy radiated: bolometric magnitude

\[ m_{bol} \]
\[ m_{bol} = m_{vis} - BC \]

Colour, colour index:

\[ CI = m_B - m_V \]

E.g.: \( B-V \)
Correcting for distance:

**Absolute brightness (magnitude)**

\[ M = -2.5 \log F_{10 \text{pc}} \]

\[ m - M = -2.5 \log \left( \frac{10 \text{pc}}{D} \right)^2 + \text{const.} \]

**Distance modulus**:

\[ m - M = 5 \log D[\text{pc}] - 5 \]

**Absolute bolometric brightness**:

\[ M_{bol} = -2.5 \log L/L_\odot + 4.72 \]
Colour: indication for effective temperature $T_{eff}$

Consider black body radiation as zeroth approximation.

Relation: spectral type - colour - $T_{eff}$

Example: Harvard - classification
Stellar Parameters

- Luminosity

- (Effective) Temperature

- Radius
  - Interferometry
  - Lunar occultation
  - Eclipsing binary
    - with $L, \text{Teff}$ from $L = 4\pi R^2 \text{Teff}$

- Mass
  - Binaries
  - Spectroscopic, eclipsing
  - Kepler's 3rd law
  - Spectroscopy: $M \sin^3 i$

- Chemical composition: Spectrum

- Rotational, magnetic fields: Spectrum
Electromagnetic spectrum

Absorption by the Earth's atmosphere

Neutral gas
Ionized gas

Absorption by the interstellar gas

Sichtbarer Bereich
Radio-Bereich

\[ B_v \propto \nu^2 T \sim \frac{1}{\lambda^2} T \]
I: neutral;  II: einfach ionisiert;  III: zweifach ionisiert