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Literature

Introductions to Inverse Problems

I.J.D. Craig and J.C. Brown, Inverse Problems in Astronomy, Adam Hilger Ltd. Bristol, 1986. (An introduction, we dont have it in our library, but I have a copy)


More comprehensive treatments are


R. Parker, Geophysical Inverse Theory, Princeton, 1994. (Standard reference for geophysicists, often quoted, should perhaps be bought by the library)

A first aid for practical problem solving is good old

Image deblurring: The phase function

Geometry for the calculation of the point spread function, $\mathcal{A}$ is the plane of the aperture with a lense in front, $\mathcal{F}$ is the focal plane at a distance $f$ from the aperture. An object is assumed infinitely away in direction of $r_\infty$ and has its geometrical image at $r_{F_0}$ in the focal plane.

Electrical wave field in focal plane $F$ from a point source at $r_\infty$ is

$$E(r_F, t) = E_0 \frac{k e^{-i\omega t}}{i2\pi f} \int_A e^{i\Phi(r_F, r_A)} d^2(r_A)$$

There are three contributions to the final phase of the field on the focal plane

$$\Phi(r_F, r_A) = \underbrace{k_A \cdot r_A}_{\text{init phase in front of lense}} + \underbrace{\Delta\phi_{\text{lense}}(r_A)}_{\text{phase change due to lense}} + \underbrace{k |r_F + f - r_A|}_{\text{phase change due to propagation from plane $\mathcal{A}$ to plane $\mathcal{F}$}}$$

Making rigorous use of the Fraunhofer approximation

$$f \gg r_A \gg r_F$$

and

$$\Delta\phi_{\text{lense}} \equiv \phi_0 - \sqrt{f^2 + r_A^2}$$

$$r_{F_0} \equiv \frac{f}{k}k_A$$

gives

$$\Phi(r_F, r_A) = \phi_0 - \frac{k}{f}(r_F - r_{F_0}) \cdot r_A$$
For a circular aperture area $A$ with radius $R$ we have

$$\int_A e^{i\Phi(r_F, r_A)} d^2(r_A) = e^{i\phi_0} \int_A e^{-i\frac{k}{f}(r_F - r_{Fo}) \cdot r_A} d^2(r_A)$$

$$= 2\pi R^2 \frac{J_1(\frac{kR}{f}|r_F - r_{Fo}|)}{\frac{kR}{f}|r_F - r_{Fo}|} e^{i\phi_0}$$

which is essentially the 2D Fourier transform of the aperture area $A$ where the wave number is $|r_F - r_{Fo}|/f$ times the wave number $k$ of the electromagnetic wave.

---

The final field is

$$E(r_F, t) = E_0 \frac{kR^2}{f} \frac{J_1(\frac{kR}{f}|r_F - r_{Fo}|)}{\frac{kR}{f}|r_F - r_{Fo}|} e^{i(\phi_0 - \frac{\pi}{2} - \omega t)}$$

---

The intensity in the focal plane

$$I(r_F) = \frac{c}{2} |E(r_F, t)|^2 =$$

$$\pi R^2 \frac{c}{2} E_0^2 \frac{1}{\pi} \left(\frac{kR}{f}\right)^2 \left(\frac{J_1(\frac{kR}{f}|r_F - r_{Fo}|)}{\frac{kR}{f}|r_F - r_{Fo}|}\right)^2 P_0$$

(power collected in aperture area)

If instead of a discrete point source ideally focussed at $r_{Fo}$, we have a distributed source of brightness, we finally have to replace

$$P_0 \rightarrow I_0(r_{Fo}) d^2 r_{Fo}$$
Image deblurring: The inverse problem

Point spread function in diffraction limit

Cross section through the point spread function \((1/\pi)(J_1(\rho)/\rho)^2\) and a Gaussian \((1/4\pi) \exp(-\rho/2)^2\) of similar shape (dashed).

The final expression is

\[
I(r_F) = \int_F \frac{1}{\pi} \left(\frac{kR}{f}\right)^2 \left(\frac{J_1\left(\frac{kR}{f} |r_F - r_{F_0}|\right)}{\frac{kR}{f} |r_F - r_{F_0}|}\right)^2 I_0(r_{F_0}) \, d^2r_{F_0}
\]

Data \quad Kernel \quad Model

- If we know \(I_0\) we can calculate what intensity \(I\) we would observe (Forward problem – straight forward integration).

- Usually we observe \(I\) and would like to know what the original distribution \(I_0\) looks like (Inverse problem – much harder to solve).
Image deblurring: A practical example
Tomography: General ray paths

Tomography aims to derive the distribution of a parameter in the interior of a domain $\Omega$ from observations of a diagnostic wave field on the boundary $\partial \Omega$.

If, e.g., the refractive index is known, the ray path $C$ from any $r_0$ to $r_1$ can be calculated and the attenuated intensity

$$I_1(r_0, r_1) = I_0 \exp \left[ - \int_{C(r_0,r_1) \cap \Omega} \kappa(r) \, dr \right]$$

can be measured. Deducing the local absorption $\kappa(r)$ from these measurements is an inverse problem – a hard one depending on the ray paths $C$. 

*Diagnostic ray path $C$ through domain $\Omega$ from $r_0$ to $r_1$*
Tomography: The X-ray transform

Choose the diagnostic wave so that refractive index is close to unity and $\kappa$ moderate. So rays are straight along their initial direction $e_\theta$

$$\mathcal{C} = \{r_0 + se_\theta \mid s \in \mathbb{R}\}$$

Let $s = s_0$ and $s_1$ be the intersections of $\mathcal{C}$ with $\partial \Omega$, then

$$\ln \left( \frac{I_0 - I_1(r_0, \theta)}{I_0} \right)_{\text{data}} = \int_{s_0}^{s_1} \kappa(r_0 + se_\theta) \, ds = \int_{\Omega} K_\theta(r_0) \kappa(r) \, d^3r$$

Kernel: $K_\theta(r_0) = \int_{s_0}^{s_1} \delta(r_0 + se_\theta - r) \, ds$

Geometry of the X-ray transform. $r_0$ are the image pixel center, the integration areas (rays) are grey-shaded.

- To investigate a 3D body, take a 2D manifold of positions $r_0$ (image) with a 1D manifold of directions (the scan directions $\theta$ should cover $[0, \pi]$)

- If the $e_\theta$ all lie in a plane the 3D X-ray transform decomposes into a set of 2D transforms which can all be solved independently.
Tomography: The Radon Transform

We are measuring the line emission from an optically thin plasma cloud with distribution function \( f(v) \) from different directions \( e_\theta \).

The intensity \( I \) at frequency \( \nu \) offset from the line center by \( \Delta \nu = \nu - \nu_0 \) is proportional to the number of particles which have a velocity component \( v = c\Delta \nu/\nu_0 \) in direction \( e_\theta \).

\[
I(\Delta \nu, \theta) = \frac{\int I d\nu}{N} \int_{v \cdot e_\theta = c\frac{\Delta \nu}{\nu_0}} f(v) \, d^2v
\]

where \( N = \int f \, d^3v \) (total number of emitting particles) and \( \int I \, d\nu \) is independent of direction \( e_\theta \) it is measured in, \( c \) is the speed of light.

Geometry of the Radon transform. The integration is over planes normal to \( e_\theta \) in velocity space.

- To get the full 3D distribution measure a 1D manifold of Doppler-shifts \( \Delta \nu \) (spectra) in a 2D manifold of directions (the scan directions \( \theta \) should cover a half sphere).
- In 2D, the X-ray and the Radon transform are actually identical except that \( e_\theta \) is rotated by \( \pi/2 \).
Radiative transfer: The transport equation

In many atmospheric problems the radiance $I_\nu$ at a frequency $\nu$ depends only on height $z$ and propagation angle $\theta$.

The radiance $I(z, \theta)$ propagating at an angle $\theta$ with respect to the vertical $\hat{z}$ is modified locally by absorption and thermal emission

$$\cos \theta \frac{d}{dz} I_\nu(z, \theta) = -\kappa_\nu(z) I_\nu(z, \theta) + \epsilon_\nu$$

(no scattering)

absorption  thermal emission

Geometry for the derivation of the radiative transport equation.

Another common approximation is local thermodynamic equilibrium

$$\epsilon_\nu = \kappa_\nu(z) B_\nu(T(z))$$

where

$$B_\nu(T(z)) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_BT(z)}\right) - 1}$$

is Planck’s function at the local temperature.
Radiative transfer: Up/downward radiances

The integration is simplified by introducing the frequency dependent optical depth

\[ \tau_{\nu}(z) = \int_{z}^{\infty} \kappa_{\nu}(z') \, dz' \text{ hence } \kappa_{\nu} \, dz = -d\tau_{\nu} \]

The optical thickness of the entire atmosphere is \( \tau_{\nu}(z = 0) \equiv \tau_{\nu}^{\text{atm}} \).

Integration for upward propagation \((\cos \theta > 0)\) gives the radiance we may observe in space

\[ I_{\nu}^{\text{spc}} = I_{\nu}^{\text{srf}} \, e^{-\tau_{\nu}^{\text{atm}} / \cos \theta} + \int_{0}^{\tau_{\nu}^{\text{atm}}} B_{\nu}(T(\tau_{\nu})) \, e^{-\nu / \cos \theta} \frac{d\tau_{\nu}}{\cos \theta} \]

Integration for downward radiation \((\cos \theta < 0)\) yields the radiance we observe on the ground when looking upwards

\[ I_{\nu}^{\text{srf}} = \int_{0}^{\tau_{\nu}^{\text{atm}}} B_{\nu}(T(\tau_{\nu})) \, e^{-\nu^{\text{atm}} - \tau_{\nu} / |\cos \theta|} \frac{d\tau_{\nu}}{|\cos \theta|} \]
The radiance from the Sun for optical $\nu$ depends on $\theta$, or, equivalently, on the relative distance $\sqrt{1 - \cos^2 \theta}$ from the Sun’s center.

$$I_{\nu}(\cos \theta) = \frac{1}{\cos \theta} \int_0^{\infty} e^{-\tau_{\nu}/\cos \theta} B_{\nu}(T(\tau_{\nu})) \, d\tau_{\nu}$$

This equation can be used to infer $B_{\nu}(T(\tau_{\nu}))$, hence $T(\tau_{\nu})$ from measurement of $I_{\nu}(\cos \theta)$.

- limb darkening $\Rightarrow$ increase of $T$ with $\tau_{\nu}$

- Kernel $\exp -\tau_{\nu}/\cos \theta$ is smooth and sensitive only where it varies with $\cos \theta$, i.e., for $\tau_{\nu} \simeq 0.1 \ldots 1.5$ (lower chromosphere).

*Kernel of the limb darkening equation for $\rho = \sqrt{1 - \cos \theta} = 0.1, 0.2, \ldots, 0.8, 0.9, 0.95$. The solution of the inversion problem $B_{\nu}(T(\tau_{\nu}))$ is drawn dashed.*
Radiative transfer: Solar limb darkening observations

Observed limb darkening on the Sun – the solar disk.

Observed limb darkening on the Sun – intensity vs. $\mu = \cos \theta$ for different wavelengths from (Stix, 1989).
Radiative transfer: Molecular absorption

At GHz and THz frequencies we observe in zenith direction if we assume optically thin conditions ($\tau_\nu \ll 1$) and neglect the galactic background

$$I_\nu = \int_0^\infty B_\nu(T(z)) \kappa_\nu(z) \, dz$$

For a line at center frequency $\nu_{nm} = (E_m - E_n)/\hbar$ for a transition from state $n \rightarrow m$ of a molecule $X$

$$\kappa_\nu \propto n_{X,n} \nu \Psi(\nu - \nu_{nm}) \left(1 - e^{-\frac{E_m - E_n}{k_B T}}\right)$$

density of line shape induced emission $X$ in state $n$

- The density $n_{X,n}$ is related to the concentration $c_X$

$$n_{X,n} = n_{\text{air}} c_X \frac{g_n e^{-\frac{E_n}{k_B T}}}{Z(T)}$$

with $g_n$ the degeneracy of state $n$ and $Z(T)$ is the partition function.

- The line shape collision dominated and well modelled by a Lorentzian line profile

$$\Psi(\nu - \nu_{nm}) = \frac{\Delta \nu_C}{(\nu - \nu_{nm})^2 + (\Delta \nu_C)^2}$$

with width

$$\Delta \nu_C \approx \Delta \nu_0 \frac{p}{p_0}$$
Radiative transfer: Trace gas inversion

Typical temperature profile of the Earth’s atmosphere, factor $F_{nm}$ and line shape of kernel at various heights.

Insertion yields the inversion problem

$$I_{\nu} = \int_{0}^{\infty} F_{nm}(T(z)) \cdot \left( \frac{p(z)}{p_0} \right)^2 \left( \frac{\nu - \nu_{nm}}{\Delta \nu_0} \right)^2 \left( \frac{p(z)}{p_0} \right)^2 c_{X}(z) \, dz$$

where the $T$ dependence is concentrated in

$$F_{nm}(T) \propto B_{\nu_{nm}}(T) \cdot \frac{E_m - E_n}{k_B T} \cdot e^{-\frac{E_n}{k_B T}} \cdot \frac{e^{-\frac{E_m}{k_B T}}}{Z(T)}$$

- $X = \text{CO}_2$ or O are well mixed so that $c_X = \text{const}$
  - solve for $F_{nm}(T(z))$, i.e. $T(z)$.

- If $T(z)$ is known, solve for $c_X(z)$ of more exotic trace gases.
Radiative transfer: Trace gas inversion kernel

**Kernel height dependence**

*Height dependence of kernel functions with increasing frequency offset.*

**Kernel differences**

*Height dependence of difference between neighbouring kernel functions.*

- Combinations of the inversion equation for different $\nu - \nu_{nm}$ may give better kernels.
Helioseismology: Fundamental properties

We assume hydrostatic equilibrium

$$\nabla p_0 = g \rho_0 \quad \text{and} \quad \nabla \cdot g = -4 \pi G \rho_0$$

where in our notation $g = \hat{r} \cdot \mathbf{g}$ is negative.

The propagation of waves in the Sun is controlled by three further parameters

- Acoustic speed: $c_s^2 = \gamma p_0 / \rho_0$
- Brunt-Väisälä frequency: $N^2 = |g| \left( \frac{\partial p_0}{\gamma p_0} - \frac{\partial \rho_0}{\rho_0} \right)$
  $$= \frac{|g|}{\gamma} \left( \frac{p_0}{\rho_0^\gamma} \right)^{-1} \frac{\partial}{\partial r} \left( \frac{p_0}{\rho_0^\gamma} \right)$$
- Atmospheric scale height $H = \frac{p_0}{|g| \rho_0} = \frac{c_s^2}{\gamma |g|}$

Variation of $N$ (solid) and $c_s k_h$ (dashed) with distance from the center of the Sun for $k_h \simeq \sqrt{l(l+1)} / r$ (Christensen-Dalsgaard, 1998).
Helioseismology: Lagrangian perturbations

The inversion equation is derived from a variational principle which involves the integration of perturbations over the whole solar volume.

→ it is advantageous to change from Eulerian variables

\[ \frac{d}{dt} \mathbf{v} = -\frac{1}{\rho} \nabla p + \mathbf{g} \]

\[ \frac{d}{dt} \rho = -\rho \nabla \cdot \mathbf{v} \quad \text{and} \quad \frac{d}{dt} p = -\gamma p \nabla \cdot \mathbf{v} \]

to Lagrangian variables:

\[ \dot{\xi} = (\gamma - 1) \mathbf{g} (\nabla \cdot \xi) + c_s^2 \nabla (\nabla \cdot \xi) + \nabla (\mathbf{g} \cdot \xi) \equiv -\mathcal{A}(\xi) \]

\[ \delta_L \rho = -\rho_0 (\nabla \cdot \xi) \quad \text{and} \quad \delta_L p = -\gamma p_0 (\nabla \cdot \xi) \]

On the Sun’s surface: \( p_0 = \delta_L p = 0 \)

\[ \mathbf{x} \]
\[ \mathbf{x}_0 \]

Relation between Eulerian and Lagrangian perturbations for fixed \( t \):

\[ \mathbf{x} = \mathbf{x}_0 + \xi(\mathbf{x}_0, t) \]

\[ \mathbf{v}(\mathbf{x}, t) = \dot{\xi}(\mathbf{x}_0, t) \]

- The velocity perturbations on the Sun’s surface \( \dot{\xi} \) can be observed. Usually their FT are considered (for plane parallel geometry):

\[ \xi(\mathbf{x}_0, t) = \sum_{k_{h,\omega}} \xi_{k_{h,\omega}}(z_0) e^{i(k_{h}x_0 - \omega t)} + c.c. \]
Helioseismology: Short wavelength approximation

To qualitatively understand the observations we simplify:

- plane parallel geometry \((r \longrightarrow z)\)
- no gravity waves (g-modes are yet undetected)

For \(H k_z \gg 1\) the Lagrangian momentum equation yields

\[
\text{div} \ddot{\xi} - c_s^2 \Delta \text{div} \xi \simeq 0
\]

For an observed mode with horizontal wavenumber \(k_h\) and frequency \(\omega\) fixed

\[
k_z(z)^2 \simeq \frac{\omega^2}{c_s^2(z)} - k_h^2
\]

Since \(c_s\) increases with depth we have reflection between the surface \(z = 0\) and some \(z_{rfl}\) inside the Sun.

Propagation paths of an acoustic wave (p-mode) in the Sun. In comparison, the propagation of gravity waves (g-mode) is dashed.

- In between the reflection points, the wave must have an integer number of nodes

\[
\int_{z_{rfl}}^{z=0} k_z(z) \, dz = (n + \alpha_{rfl} + \alpha_0)\pi
\]

where \(\alpha_{0,rfl}\) are phase corrections at the respective reflection height

(from observations: \(\alpha_{rfl} + \alpha_0 \simeq 1.45\))

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Helioseismology: Short wavelength dispersion

With $N^2 \simeq 0$, we can roughly estimate how $c_s$ increases with depth inside the convection zone:

$$\frac{1}{c_s^2} \frac{d}{dz} \frac{1}{c_s^2} = \left( \frac{1}{p_0} \frac{d}{dz} - \frac{1}{\rho_0} \frac{d}{dz} \right) = \left( \frac{1}{p_0} \frac{d}{dz} - \frac{1}{\gamma p_0} \frac{d}{dz} \right) = \left( 1 - \frac{1}{\gamma} \right) \frac{g \rho_0}{p_0} = \left( \gamma - 1 \right) \frac{g}{c_s^2}$$

Insert $c_s^2(z)$ into the equation for $k_z^2(z)$

$$k_z^2(z) \simeq \frac{\omega^2}{(\gamma - 1)g z} - k_h^2$$

Insertion of $k_z(z)$ into node number integral gives

$$\int_{z_{rfl}}^0 \sqrt{\frac{\omega^2}{(\gamma - 1)g z} - k_h^2} \, dz = k_h |z_{rfl}| \frac{\pi}{2} = (n + \alpha_{rfl} + \alpha_0) \pi$$

where $z_{rfl} = (\omega/k_h)^2/(\gamma - 1)g$. The last equation yields the observable relation between the frequency and horizontal wavenumber

$$\omega^2 = \omega_{k_h,n}^2 \simeq 2(\gamma - 1)g(n + \alpha_{rfl} + \alpha_0)k_h$$

*Observed dispersion for acoustic modes inside the Sun with $l \simeq k_h R_\odot$*
Helioseismology: The variational principle

Introduction of the Fourier transform in the Lagrangian momentum equation gives

$$\omega_{kh,n}^2 \xi_{kh,n} = A_{kh}(\xi_{kh,n})$$  \hspace{1cm} (1)

an eigenvalue problem for eigenvalues $$\omega_{kh,n}^2$$ and eigenstates $$\xi_{kh,n}$$. Since $$A_{kh}$$ is hermitian, the eigenstates span a Hilbert space with normalization

$$\int_{-\infty}^{0} \rho_0 (\xi_{kh,m}^*(z) \cdot \xi_{kh,n}(z)) \, dz = \delta_{m,n} M_{kh,n}$$  \hspace{1cm} (mode inertia)  \hspace{1cm} (2)

hence the above eigenvalue equation (1) can also be written as

$$\omega_{kh,n}^2 M_{kh,n} = \int_{-\infty}^{0} \rho_0 (\xi_{kh,n}^* \cdot A_{kh}(\xi_{kh,n})) \, dz$$  \hspace{1cm} (3)

- If the calculated $$\omega_{kh,n}^2$$ do not agree with the observed frequencies, we have to vary $$\rho_0$$ (and all other parameters accordingly) in our model:

$$\rho_0 \rightarrow \rho_0 + \delta \rho_0 \quad \text{causes} \quad p_0 \rightarrow p_0 + \delta p_0 \quad ; \quad g \rightarrow g + \delta g$$

$$A_{kh} \rightarrow A_{kh} + \delta A_{kh} \quad ; \quad \xi_{kh,n} \rightarrow \xi_{kh,n} + \delta \xi_{kh,n}$$

and finally

$$\omega_{kh,n}^2 \rightarrow \omega_{kh,n}^2 + \delta(\omega_{kh,n}^2)$$

We may vary the eigenvalue equation (1) but then we need the perturbations of the eigenstates as well. A more convenient approach is to vary (3). The procedure then is almost identical to conventional perturbation theory in quantum mechanics. In any case, the orthogonality (2) and the mode inertia $$M_{kh,n}$$ remain invariant under the variation.
Helioseismology: The inversion problem

The variation of (3) yields

$$\delta(\omega_{k,n}^2) M_{k,n} = \int_{-\infty}^{0} \rho_0 \left( \xi^*_n \cdot \delta A_{k,n} (\xi_{k,n}) \right) dz$$

Hence the variation of the eigenvalue is obtained from the variation of the operator $A_{k,n}$ with respect to the unperturbed eigenfunctions. The perturbed eigenfunctions are not required.

Next, relate the variation $\delta A_{k,n}$ to the appropriate variation $\delta \rho_0$ (Fréchet derivative $\propto K_{k,n}$). We finally obtain after renormalization:

$$\frac{\delta(\omega_{k,n}^2)}{\omega_{k,n}^2} = \int_{-\infty}^{0} K_{k,n}(\xi^*_n, \xi_{k,n}) \frac{\delta \rho_0}{\rho_0} dz$$

Kernel functions $K_{k,n}$ for an acoustic mode $k_n \propto \sqrt{l(l+1)}/R_\odot$ with $l = 10$ and eigenfunction order $n = 6$. The first subscript parameter is varied, the second fixed. Here, $u = c^2_s/\gamma$, $A^* \propto N^2/|g|$ and $Y \propto$ He abundance (Kosovichev, 1999).